ARBEITSGEMEINSCHAFT
"THE NILPOTENCE THEOREM IN STABLE HOMOTOPY THEORY"
OBERWOLFACH: 4.10. - 10.10.1998

Organizers:

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A generalized homology theory is a functor h_* from a topological category (say, the category of finite simplicial complexes and continuous maps between them) to an algebraic category (say, the category of graded abelian groups). Such a functor is required to have a number of properties familiar from ordinary homology: In particular, if a map f is homotopic to a constant map, then $h_*(f) = 0$; and if ΣX denotes the suspension of X (i.e. the double cone over X), a map $f : X \to Y$ induces a map $\Sigma f : \Sigma X \to \Sigma Y$, and $h_*(\Sigma f)$ is the same as $h_*(f)$, up to a dimension shift.

Complex cobordism theory MU_* is a generalized homology theory which is important for a number of reasons: It is conceptually interesting because of its close relation with the theory of formal groups; concrete computations with this theory are often possible; and the following striking theorem is true:

Consider a map $f : \Sigma^d X \to X$ from the d-fold suspension of X to X; we call such an f a self-map of X. Then we can iterate f; for instance, we write $f^2 := f \circ \Sigma^d f : \Sigma^{2d} X \to X$, and so on. We say that f is *nilpotent* if a sufficiently high suspension of a sufficiently high power of f is null-homotopic.

Theorem. *f is nilpotent if and only if $MU_*(f)$ is nilpotent.*

This is one form of the "nilpotence theorem"; it was conjectured around 1976 by Ravenel and proved by Devinatz, Hopkins and Smith in 1988. It describes one of the rare occasions where an important geometric property is faithfully reflected in algebra by means of a homology theory. Only one special case of this general theorem was known before, namely Nishida's nilpotence theorem (1973): Let f be a self-map of the n-sphere S^n, i.e. $f : \Sigma^d S^n = S^{n+d} \to S^n$, and let $d > 0$. Then f is nilpotent.

An important theoretical consequence of the nilpotence theorem is the "thick subcategory theorem". A full subcategory $\mathcal{C}$ of the category of CW-complexes is called *thick*, if it satisfies the following conditions:

1) If X is in $\mathcal{C}$ and Y is homotopy equivalent to X, then Y is in $\mathcal{C}$.
2) If $X \vee Y$ is in $\mathcal{C}$, then X and Y are in $\mathcal{C}$.

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3) If $X \to Y \to Z$ is a cofibre sequence and if two of the spaces X, Y, Z are in $\mathcal{C}$ so is the third.

The thick subcategory theorem gives a complete classification of the thick subcategories of the stable category of all p-local finite complexes, where p is a prime number. These subcategories admit a simple description in terms of what MU_* does to the spaces belonging to them.

This theorem is one of the tools needed to prove the "periodicity theorem", which is the second of the two main theorems to which this week is dedicated. This theorem asserts the existence of periodic self maps (which are very far from being nilpotent) and is best stated in terms of the Morava K - theories $K(n)$: for a fixed prime number p there exists a sequence of periodic cohomology theories $K(n)^*$, $n = 1, 2, ...$ which in some sense generalize complex K - theory; $K(1)^*$ is in fact closely related to complex K - theory. It is convenient to define, for each prime p, $K(0)^*$ to be equal to rational cohomology.

If X is a finite p - local CW complex and n is minimal such that $K(n)^*X \neq 0$ then for some suitable d there exists a stable self map $f : \Sigma^d X \to X$ which induces an isomorphism in $K(n)^*X$ and is trivial in $K(m)^*X$ if $m \neq n$. Moreover, up to taking sufficiently high powers (with respect to composition) this self map is stably unique.

Before this theorem was proven, only very few periodic self-maps were known. They had been discovered by Adams, Toda und Larry Smith. Each of them had led to an infinite family of nontrivial elements in the stable homotopy groups of spheres. Although the periodicity theorem has not produced so far further families, it provides some new systematic insight into the stable homotopy groups of spheres and the stable homotopy category of finite CW complexes.

Lectures

Each participant should volunteer to give one of the following lectures. Each lecture lasts 60 minutes.

1. Introduction

This talk should begin with a brief introduction to homotopy groups of spheres, with the emphasis on stable homotopy groups. After a definition of homotopy groups and stable homotopy groups the following results should be briefly discussed: $\pi_n(S^n)$, the Hopf map and $\pi_3(S^2)$, the Freudenthal suspension theorem, Serre's finiteness results, the image of the J - homomorphism, Nishida's nilpotence theorem. Computations of the stable stems in low dimensions should be shown (slide!).

Then the periodic families (the α, β and γ - families) should be presented and the periodicity and nilpotence conjecture should be stated. Finally the chromatic filtration should be introduced (as in [R3, chap. 2.5]).

References: [R1], [R2], [R3, chap.1 and 2]

2. Spectra

The category of spectra should be defined and its formal properties established, i.e. it should be shown that the homotopy category of CW spectra is a triangulated category with a compatible monoidal structure (given by the smash product). Examples (suspension spectra, Moore spectra and Eilenberg-MacLane spectra) should be presented. Then homology and cohomology theories defined by spectra should be introduced. The Brown representability theorem should be briefly discussed.

References: [A, Part III, chap. 2,3 and 4], [Sw, chap. 8,9 and 13], [R3,5.1, A1-A4].

3./4. Localization of spectra and the Adams spectral sequence

One should start by discussing localization of spectra at a prime number p. In this case localization is simply given by forming the smash product with an appropriate Moore spectrum. Then Bousfield localization with respect to a generalized homology theory should be introduced [B]. (Among many other things this is important for discussing the convergence of the Adams spectral sequence.)

Then the Adams spectral sequence for a generalized cohomology theory should be introduced: Adams resolutions, conditions for convergence and conditions for the identification of its E_2 -term as an Ext-group ([A], [B], [R2], [R3]). For the last part one needs to introduce Hopf algebroids and explain that under suitable assumptions a ring spectrum E gives rise to a Hopf algebroid (E_*, E_*E) [R2].

References: [A, part III.15], [B], [HO, A6, A7], [R2, chap. 2 and A 1.1], [R3, A6, B3]. [Sw, chap. 17]

5. The Steenrod algebra

A construction of Steenrod operations should be outlined. One can either work on the chain level (e.g. [Sw, chap. 18]) or, preferably, in a more homotopy theoretic framework (e.g. [G, chap. 27]). The basic properties should be given ([SE, chap.1]. Then the Steenrod algebra and its dual (including their Hopf algebra structures) should be introduced. The operations Q^i and P_t^s should be defined. Finally the Steenrod algebra and its dual should be identified with the cohomology resp. homology of Eilenberg-MacLane spectra. It is probably a good idea to restrict attention to the case $p = 2$.

References: [Mi], [G], [SE], [Sw]

6./7. BU **and** MU

The space BU should be introduced and its role as a classifying space of stable vector bundles explained. Then one should present the homology and cohomology of BU as a Hopf algebra. Bott periodicity should be discussed in the form of a homotopy equivalence $\Omega SU \simeq BU$ [Sw, chap. 11 and 16].

Thom spaces and Thom spectra should be introduced and the Thom isomorphism should be explained.

Then complex cobordism should be defined. The geometric definition should be briefly mentioned, but emphasis is to be on the construction via the Thom spectrum MU. $H_*(MU;\mathbb{F}_p)$ should be described as a comodule over the Steenrod algebra and the computation of $\pi_*(MU)$ via the Adams spectral sequence should be outlined.

References: [A, part II], [HO, A2], [R3, in particular B1 and B2], [Sw, chap. 20]

8. MU **and formal group laws.**

After having introduced formal group laws one should briefly explain how formal group laws are attached to a complex oriented cohomology theory. The isomorphism of $\pi_*(MU)$ with Lazard's ring, the identification of the Hurewicz homomorphism for MU in terms of the logarithm and the description of the Hopf algebroid MU_*MU in terms of formal group laws should be given.

References: [A, part II], [Ha], [R2], [R3, in particular B 4]

9./10. Brown-Peterson theory BP**. Properties of** MU_*MU **and** BP_*BP **comodules**

Firstly p - typical formal groups and BP - theory should be introduced. The Hopf algebroid (BP_*, BP_*BP) should be explained in terms of formal groups and the Araki resp. Hazewinkel generators should be defined [R2, A2]. Then invariant prime ideals in MU_* resp. BP_* and the Landweber filtration theorem should be discussed.

The second talk could then concentrate on applications: the Landweber exact functor theorem (examples: rational homology, complex K - theory and $E(n)$ - theories), the thick subcategory theorem in the category of MU_*MU resp. BP_*BP comodules. The chromatic filtration (in terms of the localization functors L_n) should be presented.

References: [L1], [L2], [L3], [R2], [R3, chap. 3, 7.5 and B5, B6], [Ru]

11./12. Morava K - theories, the nilpotence theorem in terms of Morava K - theories and the thick subcategory theorem in stable homotopy theory

The Morava K - theories $K(n)$ and their connected versions $k(n)$ should be introduced and their main properties should be recalled [HS, sect.1]. The cohomology of $k(n)$ as a module over the Steenrod algebra should be described [HS, Section 4.3].

Then Spanier-Whitehead duality should be discussed [DP], [Sw, p.321ff].

The Morava K - theoretic version of the nilpotence theorem should be stated and derived from the MU - theoretic version. [HS, sect. 2]. Finally the thick subcategory theorem in stable homotopy theory should be deduced from the Morava K - theoretic version of the nilpotence theorem as in [R3] or [HS, section 1 and 2].

References: [BM], [DP], [JW], [H], [HO], [HS], [R3, 5.2, 5.3], [Ru], [Sw]

13./14. The periodicity theorem

These two talks could be organized as follows:

In the first talk v_n-self maps should be defined and the statement of the periodicity theorem and of "uniqueness" of v_n-self maps should be given. [HS, Theorem 9 and 11]. The telescope conjecture should be briefly discussed [R3, 7.5]. (Attention: This remains still a conjecture despite of various published statements to the contrary (see e.g. [R4])). Then one should show that the subcategory of finite p-local spectra admitting a v_n-self map is a thick subcategory and finally one could prove Theorem 11 [HS, sect. 5].

The aim of the second talk should then be to outline the construction of a v_n-self map for a complex which is "strongly type n" in the sense of Definition 6.2.3 in [R3]. One should in particular introduce Margolis homology and discuss its relevance for vanishing lines in Adams spectral sequences and the vanishing of Morava K - theories [HS, sections 4.2 and 4.3]. In this talk one should probably restrict attention again to the case $p = 2$.

References: [HS, in particular section 4], [R3, in particular section 6], [R4]

15. The Smith construction

In this section a complex which is strongly of type n should be constructed using suitable idempotents in the group ring of symmetric groups.

References: [R3, 6.2, 6.4 and appendix C]

16. Proof of the nilpotence theorem I

The "smash and composition versions" of the nilpotence theorem should be deduced from the "ring spectrum version". The proof of the "ring spectrum version" should be outlined: the spectra $X(n)$ and the spectra $G(k)$ should be introduced and the first two steps in the proof should be carried out [R3, Lemma 9.1.5 and 9.1.6].

References: [HO, chap. 1,3], [R2, sect. 9], [DHS, sect. 1,2 and 4]

17. Proof of the nilpotence theorem II

$\Omega\Sigma X$ and the James construction should be presented, Hopf maps should be introduced. Then the Snaith decomposition of $\Omega^2 S^{2n+1}$ and approximations of Eilenberg-MacLane spectra need to be discussed.

References: [C] [DHS, sect. 3], [HO, chap. 2], [Hu, app. 2], [R3], [Sn]

18. Proof of the nilpotence theorem III.

The final step of the proof [R3, Lemma 9.1.7.] should be presented.

References: [DHS, sect. 3], [HO, chap. 4], [R3, sect. 9],

References

[A] J.F. Adams, *Stable Homotopy and Generalised Homology*, Lecture Notes in Mathematics, University of Chicago (1975).

[BM] N. Baas and I. Madsen, *On the realization of certain modules over Steenrod's algebra*, Math. Scand. **31** (1972), 220-224.

[B] A.K. Bousfield, *The localization of spectra with respect to homology*, Topology **14** (1979), 257-281.

[C] R.L. Cohen, *Odd primary infinite families in stable homotopy theory*, Memoirs of the AMS **242** (1981).

[DHS] E. Devinatz, M.J. Hopkins and J.H.Smith, *Nilpotence and Stable Homotopy Theory*, Annals of Math. **128** (1988), 207-241.

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[H] M.J. Hopkins, *Global Methods in Homotopy Theory I*, Proceedings of the 1985 LMS Symposium on Homotopy Theory (1987), 73-96.

[HO] E. Ossa and M.J. Hopkins, *Lectures on the Nilpotence Theorem*, in preparation (1998).

[HS] M.J. Hopkins and J.H.Smith, *Nilpotence and Stable Homotopy Theory II*, preprint, available from the hopf archive (http://hopf.math.purdue.edu/pub/hopf.html).

[Hu] D. Husemoller, *Fibre bundles*, Springer Verlag (1966).

[JW] D: Johnson and S. Wilson, *BP-operations and Morava's extraordinary K-theories*, Math. Zeit. **144** (1975), 55-75.

[L1] P. Landweber, *Annihilator ideals and primitive elements in complex cobordism*, Ill. Journal of Math. **17** (1973), 273-284.

[L2] P. Landweber, *Associated prime ideals and Hopf algebras*, Journal of Pure and Applied Algebra **3** (1973), 43-58.

[L3] P. Landweber, *Homological properties of comodules over MU_*MU and BP_*BP*, American Journal of Math. **98** (1976), 591-610.

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[R1] D. Ravenel, *Localization with respect to periodic cohomology theories*, American Journal of Mathematics **106** (1984), 351-414.

[R2] D. Ravenel, *Complex Cobordism and Stable Homotopy Groups of Spheres*, Academic Press. (1986).

[R3] D. Ravenel, *Nilpotence and Periodicity in Stable Homotopy Theory*, Ann. of Math. Studies **128** (1992).

[R4] D. Ravenel, *Life after the telescope conjecture*, Algebraic K-theory and algebraic topology (Lake Louise, AB, 1991), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci. **407** (1993), 205-222.

[Ru] Y. Rudyak, *On Thom Spectra, Orientability and Cobordism*, Springer Verlag (1998).

[Sn] V. Snaith, *A stable decomposition for $\Omega^n\Sigma^n X$*, J. London Math. Soc. **2** (1974), 577-583.

[SE] N.E. Steenrod and D.B.A. Epstein, *Cohomology Operations*, Ann. of Math. Studies **50** (1962).

[Sw] R. Switzer, *Algebraic Topology – Homotopy and Homology*, Springer Verlag (1975).

[HO] is expected to be available in time for preparation of the talks.

Errata for [R2] and [R3] can be downloaded from http://www.math.rochester.edu:8080/u/drav/

Applications should be sent, preferably via e-mail to H.-W. Henn and/or W. Singhof before August 15.

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I would like to participate in the Arbeitsgemeinschaft with the title "Nilpotencve Theorem in Stable Homotopy Theory". I am willing to give a lecture:

- First choice
- Second choice
- Third choice

Name:
Address:

e-mail: