Invitation to the Arbeitsgemeinschaft at Oberwolfach: Thurston's Geometrization of 3-Manifolds

organized by Michel Boileau, Bernhard Leeb and Jean-Pierre Otal

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Surfaces can be brought in shape in a very strong way: they all admit metrics of constant Gauß curvature according to the Riemann Uniformization Theorem. It is the amazing insight of *Thurston's Geometrisation Program* that a geometrisation in similar spirit should remain possible in dimension 3: *After cutting a 3-manifold in a canonical topological manner the resulting pieces admit geometric structures.*

Let us make this a bit more precise: The cutting procedure is a canonical topological process, namely the combination of the connected sum decomposition into prime factors and the Jaco-Shalen-Johannson decomposition by cutting along incompressible tori and annuli. The good notion of *geometric structure* on a 3-manifold is a complete locally homogeneous metric of finite volume on its interior. Whereas in dimension 2 there are just the 3 constant curvature geometries, it turns out that in dimension 3 there are 8 essentially different geometries which allow finite volume quotients: the 3 constant curvature geometries, 2 product geometries, and the twisted geometries Nil, $\widetilde{PSL}(2,\mathbb{R})$ and Sol. Seven of these geometries and their quotients are fibered in a canonical way. The remaining *hyperbolic geometry* hosts the richest zoo of quotient manifolds. The Geometrisation Conjecture can be understood as a classification statement for 3-manifolds modulo the classification of hyperbolic 3-manifolds. Two special cases of apparently quite different flavour remain wide open and are central conjectures in 3-dimensional topology: The *Generalized Poincaré Conjecture* that 3-manifolds with finite fundamental group are spherical, and the *Hyperbolization Conjecture* that compact irreducible atoroidal 3-manifolds with incompressible boundary are hyperbolic, cf. [Sc]. Thurston's *Hyperbolisation Theorem* asserts that the Hyperbolization Conjecture holds for Haken manifolds, i.e. in the presence of π_1-injective embedded surfaces. This result is one of the major results in 3-dimensional topology and gives to hyperbolic geometry and Kleinian group theory a fundamental role in 3-dimensional topology. It has been announced by Thurston in the mid 70's and provided with evidence in his Notes [Thu3] and a series of papers. Only in recent years, a number of complete accounts became available.

We will focus in this workshop on one partial but substantial aspect of the Hyperbolization Theorem, namely on the case of 3-manifolds fibering over the circle. It already contains a wealth of fundamental concepts and ideas introduced by Thurston to 3-dimensional topology, and we will try to understand it in as much detail as possible. Several areas of mathematics melt in a beautiful interaction: 3-dimensional topology, Kleinian groups, Teichmüller theory and measurable dynamics. We will largely follow the approach by Otal [O], using group actions on trees and Skoras Theorem for the proof of the Double Limit Theorem which is the heart of hyperbolization.

The **main references** for this workshop will be:

On Hyperbolization: The books by Jean-Pierre Otal [O] and Misha Kapovich [Ka], Sullivans Bourbaki talk [Su3], Thurston's Notes [Thu3] and the notes on The Notes [Ep].

On diffeomorphisms of surfaces: the book [CB] and the Orsay seminar notes [FLP].

For background on the Geometrisation Programm: Thurstons book [Thu4] and the article by Peter Scott [Sc].

Lecture 1. *Geometrisation of mapping tori.*

The purpose of this lecture is to explain the geometrisation program in the special case of mapping tori of surface diffeomorphisms. In this case only six of the eight geometries are needed for geometrisation!

The case of sphere bundles is trivial. Explain in detail the baby example of torus bundles, including: the Teichmüller space of flat structures is hyperbolic plane; the mapping class group of the torus is $SL(2,\mathbb{Z})$; the classification of torus diffeomorphisms into elliptic / parabolic / hyperbolic types according to their action on Teichmüller space, and the existence of geometric structures modelled on Euclidean resp. Nil-, resp. Solv-geometry.

In the higher genus case, discuss the Nielsen-Thurston classification of surface diffeomorphisms and its relation to the Jaco-Shalen-Johannson decomposition of the mapping torus.

References: [Sc], [Thu4], [CB], [FLP], [Thu1], [Thu2].

Lecture 2. *Survey of the proof of the hyperbolization theorem.*

Reserved.

Lecture 3. *Hyperbolic geometry.*

Models for hyperbolic plane / space, in particular the Poincaré ball model and the hyperboloid model. Busemann functions and horoballs.

The boundary at infinity and its conformal structure; isometries of hyperbolic space correspond to conformal diffeomorphisms at infinity.

Classification of isometries into elliptic / parabolic / hyperbolic (loxodromic) ones according to their fixed points and dynamics on the compactification.

(A little bit of) geometry of geodesic triangles, construction of hyperbolic structures on orientable surfaces of higher genus.

Quasi-geodesics and Morse Lemma. Quasi-isometries extend continuously to quasiconformal homeomorphisms of the sphere at infinity.

References: Ch. 4 in [Ka], [MT], [Thu4], [Thu3], [BP], [Ra].

Lecture 4. *Kleinian groups*

Definition. Holonomy of a hyperbolic structure, correspondence between hyperbolic manifolds and torsion-free Kleinian groups.

Associated conformal dynamical system: action on the sphere at infinity, limit set, domain of discontinuity, Kleinian manifold.

Convex hull of the limit set / Nielsen convex core, convex-cocompact and geometrically finite groups.

Margulis Lemma / Zassenhaus Lemma, structure of the thin part, in particular of cusps.

Variety of representations and its "algebraic" topology. A sequence of bounded representations of a non-elementary group remains bounded if the traces of all elements remain bounded. Closedness of discrete and faithful representations in the representation variety (as a consequence of the Margulis Lemma).

References: ch. 6 and 9 in [Ka] and references therein, [Ep], [MT], [CS], [Bea], see also [An].

Lecture 5. *Explicit examples of hyperbolic 3-manifolds fibering over the circle.*

Gieseking manifold, right-angled dodecahedral manifold, figure 8 knot complement.

References: Ch. 9 of [O] and references there, [Su3] ch. 4 and 6 in [Thu3], [Ri], [Mi].

Lecture 6. *Ahlfors Lecture.*

Description and outlines of proofs for Ahlfors' fundamental results about Kleinian groups acting on hyperbolic 3-space: Ahlfors Lemma (cf. 1.1.5 in [O]), Ahlfors Finiteness Theorem, Ahlfors Measure Zero Theorem for geometrically finite Kleinian groups.

References: Ch. 1 of [O] and references therein, [MT], [Mo], [Ah1].

Lecture 7. *Geodesic laminations on hyperbolic surfaces.*

Geodesic laminations. Geometry of complementary regions, Hausdorff topology on the space of geodesic laminations, compactness.

Measured geodesic laminations. Intersection number: continuity. Train tracks: definition, application: construction of (arational) measured geodesic laminations, charts for the space of projectivised measured laminations, density of measured laminations with compact leaves.

References: Appendix and ch. 3 in [O], [CB].

Lecture 8. *Actions on trees and degenerations of actions on negatively curved spaces.*

Definition(s) of real trees / metric trees, characterization as spaces of infinite negative curvature resp. 0-hyperbolic spaces. Classification of isometries on trees into elliptic and hyperbolic ones according to their minimal displacement being zero or positive. Definition of minimal and small actions.

Ultrafilters and ultralimits of metric spaces and group actions. Proof that the passage to ultralimits preserves curvature bounds. Degeneration of group actions on negatively curved spaces (in particular hyperbolic spaces) to small actions on metric trees; the smallness is a reflection of the Margulis Lemma.

Compactness criterion for representation varieties: the non-existence of small actions on trees implies compactness.

References: Ch. 10 and 11 in [Ka], also: ch. 2 in [O], [Pau1], [CM].

Lecture 9. *Degenerations of surface group actions.*

Description of an important example of surface group actions on trees: "geometric actions", i.e. the dual tree for a measured geodesic lamination on a surface. State Skoras Theorem that small actions of a surface group on a tree are geometric. Realisation of laminations (ch. 3 in [O]). Realisation of train tracks (ch. 4 in [O]). Uniform growth property of translation lengths (Thm 4.0.1 in [O]).

References: Ch 3+4 in [O], ch. 12 in [Ka].

Lecture 10. *Teichmüller space.*

Definition of Teichmüller space as conjugacy classes of discrete faithful representations into $PSL(2,\mathbb{R})$. It is a connected component of the representation variety.

Fenchel-Nielsen coordinates, proof that Teichmüller space is a ball.

Embedding of Teichmüller space into projective space via projectivized length spectrum.

Mapping class group: Definitions as outer automorphisms resp. isotopy classes of diffeomorphisms, discontinuity of the action on Teichmüller space.

References: Ch. 1 in [O], [FLP] (in particular the exposé by Douady).

Lecture 11. *Thurston's compactification of Teichmüller space and the dynamics of pseudo-Anosov diffeomorphisms.*

Thurston's compactification of Teichmüller space, extension of the action of the mapping class group (naturality).

Interpret / describe the ideal boundary points as (i) projectivized measured laminations (ii) small actions on trees. (Use the continuous embedding of the space of projectivized measured geodesic laminations into the boundary of Teichmüller space via intersection numbers; the surjectivity follows from Skora theorem.)

Explain the Nielsen-Thurston classification of surface diffeomorphisms in terms of their dynamics on the ideal boundary of Teichmüller space. Charaterize pseudo-Anosov diffeomorphisms by their dynamics.

References: [CB], [Thu1], ch. 12 in [Ka], [FLP].

Lecture 12. *Quasiconformal homeomorphisms in dimension 2.*

Definitions: analytic definition via excentricity (explain smooth case first), definition via modulus of quadrilaterals.

Some regularity properties, mainly: absolute continuity, a.e. differentiability.

Compactness theorem for quasiconformal homeomorphisms with bounded distortion.

Beltrami differential equation.

References: [Ah2], [Ga], [LV].

Lecture 13. *Teichmüller space from an analytical view point.*

Recall the correspondence between conformal and hyperbolic structures on surfaces via the Uniformization Theorem.

Teichmüller metric: completeness as a consequence of the compactness theorem for quasiconformal homeomorphisms with uniformly bounded conformal distortion.

Description of extremal quasiconformal maps in a given homotopy class using singular Euclidean structures resp. holomorphic quadratic differentials. First describe the baby example 2-torus where the extremal maps are affine (with respect to Euclidean structures in the conformal class).

Dynamics of pseudo-Anosov diffeomorphisms on the space of projectivized measured geodesic laminations: proof that there exist 2 fixed points, an attractive and a repulsive one.

References: [Ab], [Ga], [Le], [N], [Be], [Thu1], [HM1], [HM2].

Lecture 14. *Quasifuchsian space.*

Definition of quasifuchsian groups as Kleinian groups quasiconformally conjugate to Fuchsian groups. Their convex cocompactness / geometric finiteness.

Ahlfors-Bers Theorem: unique solvability of the Beltrami equation; application to the construction of quasiconformal homeomorphisms, and quasiconformal conjugation of Kleinian groups. Parametrisation of quasifuchsian groups by conformal structures on the boundary at infinity.

Mention quasiconformal stability (openness of quasifuchsian representations in the representation variety).

References: Ch. 1 in [O], [Ah2], [An].

Lecture 15. *The Double Limit Theorem.*

This lecture concerns Theorem 5.0.1 in [O] which is at the heart of Hyperbolization. It says that certain diverging sequences of quasifuchsian actions stay bounded in the representation variety and hence provides points in the closure of quasifuchsian space. Non-trivial representations of this sort are needed since they arise as holonomy representations of the fiber of hyperbolized mapping tori.

The proof will be based on the Ahlfors Lemma (cf. lecture 6), Skoras Theorem (cf. lecture 9), and a uniform growth property for translation lengths (Thm 4.0.1 in [O], cf. Lecture 9).

References: Ch. 5 in [O].

Lecture 16. *Proof of the Hyperbolization Theorem.*

The aim of this lecture is to complete the proof of the Hyperbolization Theorem modulo Sullivan's results about quasiconformal rigidity (cf. lectures 17 and 18). The goal is to build a discrete faithful representation of the fundamental group of the mapping torus M_ϕ into $PSL(2,\mathbb{C}) \cong Isom^+(H^3)$ which amounts to constructing a representation $\pi_1(F) \to PSL(2,\mathbb{C})$ of the fundamental group of the fiber whose conjugacy class is fixed under the action of the pseudo-Anosov ϕ on the representation variety. Once this is done, one can conclude using a classical result of Stallings in 3-dimensional topology. The desired surface group representation is necessarily "complicated" in the sense that it can't possibly be quasi-fuchsian. It will be found in the boundary of quasi-fuchsian space, and the main portion of this talk is devoted to the study of representations in the boundary of quasifuchsian space provided by the Double Limit Theorem.

The approach is indirect using Sullivan's Theorem. The difficult step is to prove that the limit set of such a representation is the full 2-sphere at infinity.

References: Ch. 6 in [O].

Lectures 17. *Measurable dynamics and the Sullivan-Mostow rigidity theorem I.*

(i) The splitting of an absolutely continuous action of a countable group into conservative and dissipative part. This is a general result in ergodic theory, which can be regarded as the measurable version of the splitting of a topological action into non-wandering / wandering parts. Klaus Schmidt Lemma.

(ii) Specializing these results to Kleinian groups (in arbitrary dimension, not necessarily finitely generated) leads to a theorem of Sullivan. It establishes ergodic properties for the conservative part of the action of a Kleinian group: There is no measurable group invariant plane field supported on a set of positive measure in the conservative part.

References: Ch. 7.1 and 7.2 in [O] and references therein, Sullivan's papers, [MT].

Lectures 18. *Measurable dynamics and the Sullivan-Mostow rigidity theorem II.*

We now restrict to a finitely generated Kleinian group acting on hyperbolic 3-space, respectively, on the 2-sphere at infinity. The aim of this lecture is to prove that in this situation, the topological and measurable decompositions of the dynamical system coincide, i.e. the limit set contains no dissipative part for the group action. A consequence is a strengthening of the Mostow rigidity theorem: If the limit set of a finitely generated Kleinian group is the full 2-sphere then any quasi-conformally conjugate Kleinian group is actually conformally conjugate.

References: Ch. 7.3 in [O], Sullivans papers, [MT].

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I would like to take part in the Arbeitsgemeinschaft on *Thurston's Geometrization of 3-Manifolds* and give a lecture, preferably No. ... or No. ... .

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