McKay Correspondence + Overview of the Seminar

21.11.2011

$SL_2(\mathbb{C}) \supseteq G$ finite subgroup

John McKay 1979

$$\text{(I)}\ \{\text{Representations of } G\} \xleftrightarrow{\ \text{III}\ } \{\text{Geometry of } \widetilde{\mathbb{C}^2/G}\}\ \text{(II)}$$

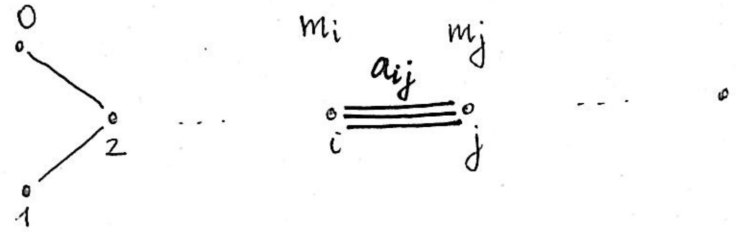
(I) Irreducible representations of G over $\mathbb{C}$ $= \mathrm{IrrRep}(G) = \{V_0, V_1, \dots, V_n\}$, $V_0 = \mathbb{C}$ trivial

- $m_i = \dim_{\mathbb{C}}(V_i)$
- $SL_2(\mathbb{C}) \supseteq G \circlearrowright \mathbb{C}^2 = V$ fundamental representation
- $V_i \overset{\mathbb{C}}{\otimes} V \cong \bigoplus_{j=0}^{n} V_j^{a_{ij}}$

Fact: $a_{ij} = a_{ji}$ $\forall\, 0 \le i, j \le n$

Def McKay graph of G

$MK'(G)$



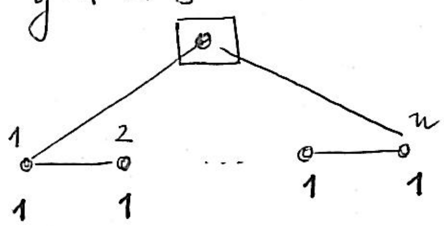
$$MK(G) = MK'(G) \setminus \{0\}$$

Example $\xi = e^{\frac{2\pi i}{n+1}}$ $\quad G = \left\langle \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix} \right\rangle \cong \mathbb{Z}/(n+1)\mathbb{Z}$

G is abelian $\Longrightarrow$ all irreducible representations of G are 1-dimensional

$\mathrm{IrrRep}(G) = \{V_0, V_1, \dots, V_n\}$
- $V_\ell = \mathbb{C}$
- $g \cdot 1 = \xi^{-\ell}$

$V \cong V_1 \oplus V_n \Longrightarrow MK(G) =$



<u>Remark</u> All finite subgroups of $SL_2(\mathbb{C})$ are known

$$\langle a, b, c \mid a^p = b^q = c^r = abc \rangle \qquad \frac{1}{p} + \frac{1}{q} + \frac{1}{r} \geq 1$$

(II) $SL_2(\mathbb{C}) \supset G$ operates on $R = \mathbb{C}[|x,y|]$

$$\underset{g}{\cup} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \begin{aligned} x &\overset{g}{\mapsto} ax + by \\ y &\overset{g}{\mapsto} cx + dy \end{aligned} \qquad p(x,y) \overset{g}{\mapsto} p(gx, gy)$$

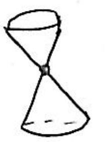
$A = R^G$ ring of invariants

<u>Theorem</u>: A is a hypersurface singularity

<u>Example</u> $\xi = e^{\frac{2\pi i}{n+1}}$ $\qquad G = \left\langle g = \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix} \right\rangle$

$$x^p y^q \xrightarrow{g} \xi^{p-q} x^p y^q$$

$$\Rightarrow R^G = \mathbb{C}[|\underset{u}{\overset{\|}{x^{n+1}}}, \underset{w}{\overset{\|}{xy}}, \underset{v}{\overset{\|}{y^{n+1}}}|] \cong \mathbb{C}[|u,v,w|]/(uv - w^{n+1}) \quad (A_n\text{- singularity})$$



$n = 1 \qquad uv - w^2 = 0$

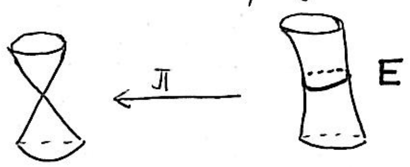
Let $S = \mathrm{Spec}(A) \xleftarrow{\pi} X$ minimal resolution of singularities

$\quad\ \ \cup \qquad\qquad\quad\ \ \cup$

$\quad\ \ 0 \quad \longleftarrow \quad E$ exceptional divisor

<u>Theorem</u> E is a tree of projective lines. $E = \bigcup_{i=1}^{n} E_i \qquad E_i \cong \mathbb{P}^1$

<u>Example</u> A_1



$E \cong \mathbb{P}^1 \qquad S^1_{\mathbb{R}} \cong \mathbb{RP}^1$

$A_n \qquad E =$  chain of n $\mathbb{P}^1$-s

<u>Def</u>: Intersection matrix

$$C \in \mathrm{Mat}_{n \times n}(\mathbb{Z}) \qquad c_{ij} = \begin{cases} \#(E_i \cap E_j) & 1 \leq i \neq j \leq n \\ -2 & i = j \end{cases}$$

Dual intersection graph Γ:

vertices $= \{1, 2, \dots, n\}$

$i \neq j$ are connected by c_{ij} edges

Fundamental cycle

$$Z = \text{minimal} \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} \in \mathbb{Z}^n_{>0} \ : \ \underset{\underset{(0,\dots,0,1,0,\dots,0)}{\parallel}}{e_i^{tr}} \cdot C \cdot Z \leq 0 \quad \forall\, 1 \leq i \leq n$$

$$\parallel$$

$$\sum_{i=1}^n m_i [E_i]$$

<u>Example</u>: $A = \mathbb{C}[[x,y]]^{\mathbb{Z}/(n+1)\mathbb{Z}} \cong \mathbb{C}[[u,v,w]]/uv - w^{n+1}$ A_n-singularity

$$\Rightarrow \quad C = \begin{pmatrix} -2 & 1 & & 0 \\ 1 & -2 & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & 1 & -2 \end{pmatrix} \qquad Z = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \qquad \Gamma = \underset{1}{\overset{1}{\circ}} - \underset{1}{\overset{1}{\circ}} \ \cdots \ \underset{1}{\overset{1}{\circ}}$$

McKay's Observation:

For any finite subgroup $G \subseteq SL_2(\mathbb{C})$

$$\exists \quad MK(G) \xrightarrow{\sim} \Gamma$$

$$\left\{ \begin{array}{ccc} \mathrm{IrrRep}(G) \setminus \{\mathrm{triv}\} & \longleftrightarrow & \mathrm{IrrComp}(E) \\ a_{ij} & & c_{ij} \\ \dim(V_i) & & m_i \end{array} \right.$$

<u>Explanation</u> (~ 1985 Auslander, Gonzalez-Springer, Verdier, Artin, ...)

$\exists$ an intermediate object: the stable category of maximal Cohen-Macaulay modules over $A = \mathbb{C}[[x,y]]^G$

$$\begin{tikzcd} \mathrm{IrrRep}(G) \setminus \{\mathrm{triv}\} \arrow[rr, leftrightarrow, "\sim"] \arrow[dr, leftrightarrow, "\text{algebraic McKay C}"'] & & \Gamma_{(X,E)} \arrow[dl, leftrightarrow, "\text{geometric McKay Corr}"] \\ & \mathrm{Ind}(\underline{\mathrm{CM}}(A)) & \end{tikzcd}$$

Overview of the Seminar

$(A, \mathfrak{m})$ normal surface singularity, $A/\mathfrak{m} =: k = \bar{k}$ $\quad [\operatorname{char}(k) = 0]$

Theorem (Serre)

$$\operatorname{Ref}(A) := \{ M \in A\text{-mod} \mid M \xrightarrow{\sim} M^{\vee\vee} \} = \operatorname{CM}(A) = \{ M \in A\text{-mod} \mid \operatorname{Hom}_A(k, M) = 0 = \operatorname{Ext}^1_A(k, M) \}$$

$$\begin{array}{ccccl} S = \operatorname{Spec}(A) & \xleftarrow{\pi} & X & & \text{minimal resolution of singularities} \\ \cup & & \vee\!\mid & & \\ \{\mathfrak{m}\} = o & \longleftarrow & E & & \text{exceptional divisor (projective curve)} \end{array}$$

Def: $\operatorname{VB}(X) \supset \operatorname{VB}^f(X) = \left\{ \mathcal{F} \;\middle|\; \begin{array}{l} \operatorname{cok}(H^0(\mathcal{F}) \otimes \mathcal{O} \xrightarrow{ev} \mathcal{F}) \text{ has finite length} \\ H^0(X, \mathcal{F}) \xrightarrow{\sim} H^0(\check{X}, \mathcal{F}) \qquad \check{X} = X \setminus E \end{array} \right\}$

full v.b.

Theorem $\quad \operatorname{CM}(A) \underset{\pi_*}{\overset{\pi^\#}{\rightleftarrows}} \operatorname{VB}^f(X)$ are quasi-inverse equivalences of categories

$$\pi^\#(M) = \pi^*(M)^{\vee\vee}$$

$$A \underset{\pi_*}{\overset{\pi^\#}{\rightleftarrows}} \mathcal{O}$$

Corollary "stable category" $\quad \underline{\operatorname{CM}}(A) \xrightarrow{\sim} \underline{\operatorname{VB}}^f(X)$

$$\operatorname{Ob}(\underline{\operatorname{CM}}(A)) = \operatorname{Ob}(\operatorname{CM}(A))$$

$$\underline{\operatorname{Hom}}_A(M, N) = \operatorname{Hom}_A(M,N) / \mathcal{P}(M,N) \qquad \mathcal{P}(M,N) = \left\{ \begin{tikzcd} M \arrow[rr, "f"] \arrow[dr, dashed] & & N \\ & A^p \arrow[ur, dashed] & \end{tikzcd} \right\}$$

Let $A = k[[x,y]]^G \qquad G \subseteq SL_2(k)$

Theorem $\underbrace{\text{[Du Val]}}$ A is a rational double point:

- $H^1(\mathcal{O}_X) = 0$
- $Z^2 = -2$

$$\Rightarrow A \cong k[[u,v,w]]/(f) \qquad f = \begin{cases} u^2 + v^{n+1} + w^2 & A_n \\ u^2 v + v^{n-1} + w^2 & D_n \\ E_6, E_7, E_8 & \end{cases}$$

Assume $k = \mathbb{C}$

Exponential sequence $\quad 0 \to \mathbb{Z}_X \to \mathcal{O}_X \xrightarrow{\exp(2\pi i -)} \mathcal{O}_X^* \to 1$

$$\begin{tikzcd} X \arrow[rr, "\pi"] \arrow[dr, "\tilde{\Gamma}"'] & & S \arrow[dl, "\Gamma"] \\ & \operatorname{Spec}(k) & \end{tikzcd} \quad \text{exact}$$

$$\Rightarrow H^1(\mathcal{O}_X) \longrightarrow H^1(\mathcal{O}_X^*) \xrightarrow{c_1} H^2(X, \mathbb{Z}) \longrightarrow H^2(\mathcal{O}_X) = \Gamma(R^2\pi_*\mathcal{O}_X)$$

rationality: $H^1(\mathcal{O}_X) = 0$; $H^1(\mathcal{O}_X^*) \cong \operatorname{Pic}(X)$; c_1 first Chern class; $H^2(X,\mathbb{Z}) \cong \mathbb{Z}^n$; $H^2(\mathcal{O}_X) = 0$; $R^2\pi_*\mathcal{O}_X = 0$

Theorem (Artin–Verdier) Let $(A, \mathfrak{m})$ be a rational double point

- $\underline{VB}^f(X) \ni \mathcal{F}$ is determined by $c_1(\mathcal{F}) \in H^2(X, \mathbb{Z})$
- $\mathrm{rk}(\mathcal{F}) = c_1(\mathcal{F}) \circ Z = \int_Z c_1(\mathcal{F})$
- $\mathcal{F}$ is indecomposable $\Rightarrow \exists!\ 1 \le i \le n : \quad c_1(\mathcal{F}) = E_i^* . \qquad E_i^* \cdot E_j = \delta_{ij}$

(A, m) simple elliptic cusp $\overset{\text{def}}{\Longrightarrow}$ • $H^1(\mathcal{O}_X) = k$ • A is Gorenstein $\qquad E = \left\{ \begin{array}{ll} 0 & \delta = 1 \\ & \delta < 1 \end{array} \right.$

Example $x^p + y^q + z^r - \lambda xyz \qquad \delta = \frac{1}{p} + \frac{1}{q} + \frac{1}{r} \le 1$

Theorem (Kahn) $\quad \underline{VB}^f(X) \xrightarrow{R_E} \underline{VB}^f(E) = \left\{ \mathcal{F} \;\middle|\; \begin{array}{l} \mathrm{cok}(H^0(\mathcal{F}) \otimes \mathcal{O} \xrightarrow{ev} \mathcal{F}) \text{ has finite length} \\ H^1(\mathcal{F}) = 0 \end{array} \right\}$

- R_E is dense (ess. surjective)
- $R_E(\mathcal{F}) \cong R_E(\mathcal{G}) \iff \mathcal{F} \cong \mathcal{G}$
- $\mathcal{F}$ is indecomposable $\iff R_E(\mathcal{F})$ is indecomposable $\qquad \underline{CM}(A) \longrightarrow \underline{VB}^f(X) \longrightarrow \underline{VB}^f(E)$
- $\mathrm{rk}(M) = \mathrm{rk}(R_E(\mathcal{F})) + h^0(\overset{R_E}{\mathcal{F}}(E))$, $\quad \mathcal{F}(E) = \mathcal{F} \otimes \mathcal{O}_X(E)|_E$

<u>Plan of my course</u>:

(3) vector bundles on 0 after Atiyah
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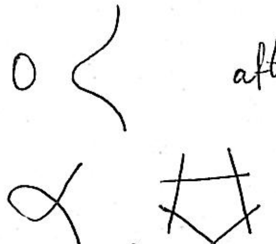
(4)



(2) classifications of CM modules and matrix problems

(5) CM modules over degenerate cusps

- $x^p + y^q + xyz$
- xyz

<u>Exercise</u> $n \geq 2$ $\xi = e^{\frac{\pi i}{n}}$

$$\sigma = \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix} \qquad \tau = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad G = \langle \sigma, \tau \rangle \subseteq SL_2(\mathbb{C})$$

(1) Show: $k[[x,y]]^G \cong k[[u,v,w]]/u^2 + wv^2 + w^{n+1}$ $\quad D_{n+2}$-singularity

(2) Compute the McKay graph of G

<u>Hints</u>

- $G \cong \left\langle \sigma, \tau \,\middle|\, \begin{array}{l} \sigma^n = \tau^2 \\ \tau^4 = 1 \end{array} \quad \tau\sigma\tau^{-1} = \sigma^{-1} \right\rangle = \mathbb{D}_n$ binary dihedral group
- $|\mathbb{D}_{2n}| = 4n$
- describe 1-dim. represent. of G
- $\sigma = \begin{pmatrix} \xi^t & 0 \\ 0 & \xi^{-t} \end{pmatrix} \qquad \tau = \begin{pmatrix} 0 & 1 \\ (-1)^t & 0 \end{pmatrix} \qquad 1 \leq t \leq n-1$

  2-dim. irreducible represent.