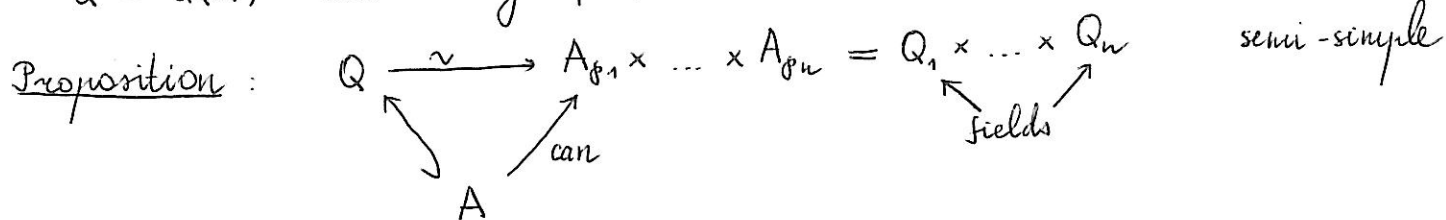


§ I Preliminaries from commutative algebra

① A reduced Noetherian ring [no embedded associated primes] is actually sufficient
 $\{p_1, \dots, p_n\} = \{p \in \text{Spec}(A) \mid \text{ht}(p) = 0\}$

$Q = Q(A)$ total ring of fractions



② Let $A \subset R \subset Q$ (finite)

Def $I := \text{Ann}_A(R/A)$ conductor ideal

Lemma $I = \{a \in A \mid aR \subseteq A\}$
 $I = IA = IR$

Lemma $I \xrightarrow{\sim} \text{Hom}_A(R, A)$
 $\downarrow \psi$
 $a \longmapsto (r \mapsto ar)$

Proof

Corollary $\forall p \in \text{Spec}(A) \quad I_p \xrightarrow[\sim]{\text{can}} \text{Ann}_{A_p}(R_p/A_p)$

Proof

Corollary $A/I =: \bar{A} \hookrightarrow \bar{R} := R/I$

Then \bar{A} and \bar{R} are supported at those p : $A_p \neq R_p$.

③ Let $M \in A\text{-mod}$

rational envelope $Q_A(M) := Q_A \otimes M \cong Q_1^{p_1} \oplus \dots \oplus Q_n^{p_n}$

$\underline{rk}(M) := (p_1, \dots, p_n)$ multi-rank of M

Def: $\text{tor}_A(M) := \ker(M \longrightarrow Q_A \otimes M)$ torsion part of M

$$\tilde{M} := R \otimes_A M / \text{tor}_R(R \otimes_A M)$$

Proposition

(i) $\forall N \in R\text{-mod}$ we have: $\text{tor}_R(N) = \text{tor}_A(N)$ as A -modules

(ii) $\forall M \in \text{TF}(A)$ $M \longrightarrow \tilde{M}$ is injective

$IM \longrightarrow I\tilde{M}$ is isom.

Proof

$$\begin{array}{ccc} \text{(i)} & Q_A \otimes N \xrightarrow{\sim} Q_R \otimes N = Q(N) & \\ & \uparrow & \uparrow \\ & N & N \end{array} \Rightarrow \text{tor}_A(N) = \text{tor}_R(N)$$

$$\text{(ii)} \quad M \longrightarrow R \otimes_A M \longrightarrow R \otimes_A M / \text{tor}(-) \stackrel{\parallel}{=} \tilde{M} \quad \text{induces}$$

$$Q(M) \xrightarrow{\cong} Q(\tilde{M})$$

Let $K = \ker(M \longrightarrow \tilde{M})$. Then

- $K \in \text{TF}(A) \Rightarrow K \subseteq Q(K) \stackrel{\parallel}{=} 0 \Rightarrow K = 0$
- $\underline{rk}(K) = 0$

Next $IM \longrightarrow I\tilde{M}$ is injective $\overset{I}{\psi}$

surjectivity: $a \begin{bmatrix} m \otimes b \\ \uparrow \\ M \quad R \end{bmatrix} = [m \otimes ab] = \overset{\psi}{ab} [m \otimes 1]$

II Main Construction

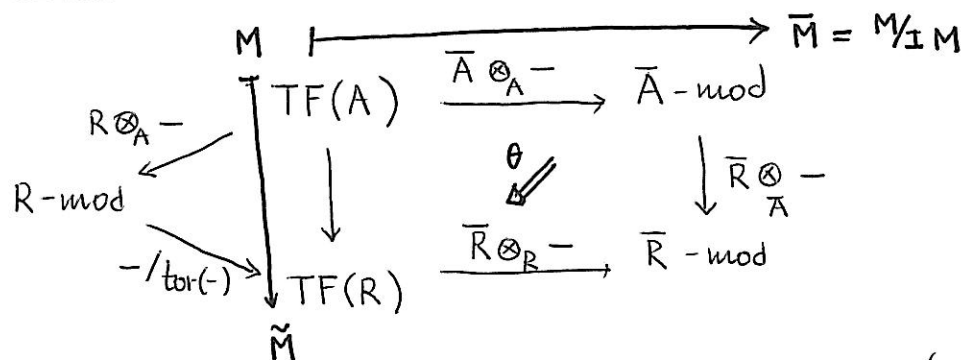
(A, m) CM curve singularity [e.g. reduced]

$$A \subset R \subset Q$$

finite

$$I = \text{Ann}_A(R/A)$$

Observe: $\bar{A} = A/I$ and $\bar{R} = R/I$ are artinian, $\underline{\text{TF}}(A) = \text{CM}(A)$



A is reduced
[or weaker: no embedded associated primes]

$$\theta_M: \bar{R} \otimes_{\bar{A}} \bar{A} \otimes_A M \xrightarrow{\sim} \bar{R} \otimes_{\bar{R}} R \otimes_A M \longrightarrow \bar{R} \otimes_{\bar{R}} \left(\begin{array}{c} R \otimes_A M \\ -/\text{tor}(-) \\ \tilde{M} \end{array} \right)$$

Idea: recover M from the triple $(\tilde{M}, \bar{M}, \theta_M)$

Note:

$$\begin{array}{ccccccc}
 0 & \rightarrow & I M & \rightarrow & M & \rightarrow & \bar{A} \otimes_A M \rightarrow 0 \\
 & & \parallel & & \downarrow & & \downarrow \tilde{\theta}_M \\
 0 & \rightarrow & I \tilde{M} & \rightarrow & \tilde{M} & \rightarrow & \bar{R} \otimes_{\bar{R}} \tilde{M} \rightarrow 0
 \end{array}$$

Snake Lemma: $\tilde{\theta}_M$ is injective.

- first version: Drozd-Roiter 1967
- current exposition [BD, "CM on non-isol. surf. sing", Appendix]

Def: category of triples $\text{Tri}(A)$

- objects: triples (\tilde{M}, V, θ)

$$\begin{array}{l} \tilde{M} \in \text{TF}(R) \\ V \in \bar{A}\text{-mod} \end{array}$$

$$\begin{array}{ccc}
 \bar{R} \otimes_{\bar{A}} V & \xrightarrow{\theta} & \bar{R} \otimes_{\bar{R}} \tilde{M} \\
 \downarrow \text{can} & \nearrow \tilde{\theta} & \\
 V & &
 \end{array}$$

gluing map (in $\bar{R}\text{-mod}$)

$\tilde{\theta}$ is injective

morphisms

$$\begin{array}{ccc}
 (\tilde{M}, V, \theta) & \xrightarrow{(\varphi, \psi)} & (\tilde{M}', V', \theta') \\
 \downarrow \varphi & & \downarrow \psi \\
 V & \xrightarrow{\bar{R} \otimes_A} & \bar{R} \otimes_R \tilde{M} \\
 \downarrow \psi & \downarrow 1 \otimes \varphi & \downarrow 1 \otimes \psi \\
 V' & \xrightarrow{\bar{R} \otimes_A} & \bar{R} \otimes_R \tilde{M}'
 \end{array}$$

Theorem: $TF(A) \xrightarrow{F} Tri(A)$ is an equivalence of categories.
 $\Psi: M \mapsto (\tilde{M}, \bar{M}, \theta_M)$
 $R \otimes_A M / \text{tors} \cong \bar{R} \otimes_A M$

Proof: quasi-inverse equivalence

$$\begin{array}{ccc}
 G: Tri(A) & \xrightarrow{\quad} & TF(A) \\
 \downarrow \Psi & & \downarrow \Psi \\
 (\tilde{M}, V, \theta) & \xrightarrow{\quad} & \tilde{M}
 \end{array}$$

Commutative diagram showing the relationship between the categories and the objects.

- $\tilde{\theta}$ is monom \implies *snake Lemma* $\implies i$ is monom $\implies M$ is torsion free
- functoriality of pull-backs $\implies G$ is a functor $\downarrow 0 \rightarrow N \rightarrow \tilde{M} \oplus V \rightarrow \hat{M} \rightarrow 0$

$$\begin{array}{ccccccc}
 0 & \rightarrow & IM & \rightarrow & M & \rightarrow & \bar{A} \otimes_A M \rightarrow 0 \\
 & & \parallel & & \downarrow & & \downarrow \tilde{\theta}_M \\
 0 & \rightarrow & I\tilde{M} & \rightarrow & \tilde{M} & \rightarrow & \bar{R} \otimes_R \tilde{M} \rightarrow 0
 \end{array}$$

$\implies GF(M) \xrightarrow{\sim} M \quad GF \xrightarrow{\quad} \mathbb{1}_{TF(A)}$

$\implies F$ is faithful.

Lemma Let $(\tilde{M}, V, \theta) \xrightarrow{(\varphi, \psi)} (\tilde{M}', V', \theta')$ be a morphism in $Tri(A)$.

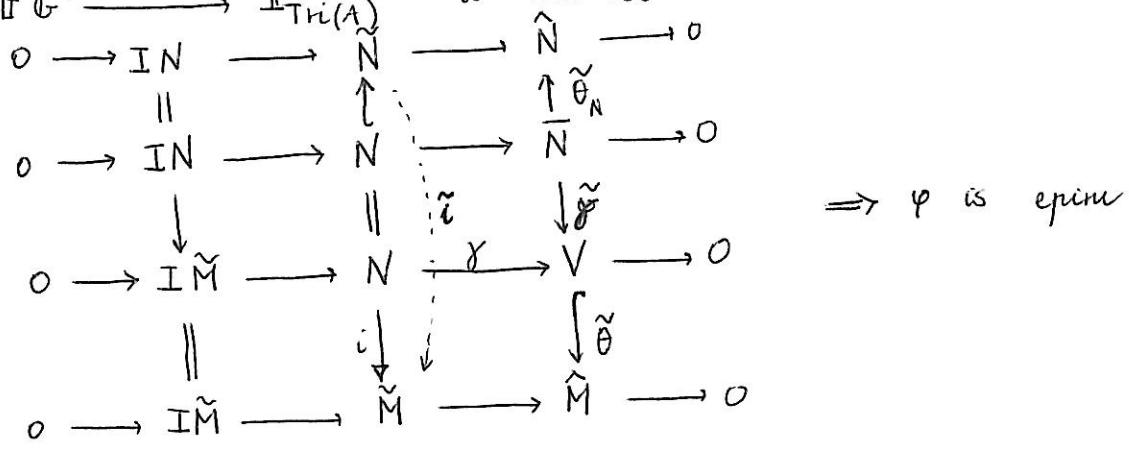
Then $\varphi = 0 \implies \psi = 0$

Proof

$$\begin{array}{ccc}
 V & \xrightarrow{\quad} & \bar{R} \otimes_A V \xrightarrow{\theta} \bar{R} \otimes_R \tilde{M} \\
 \downarrow \varphi & & \downarrow 1 \otimes \varphi \\
 V' & \xrightarrow{\quad} & \bar{R} \otimes_A V' \xrightarrow{\theta'} \bar{R} \otimes_R \tilde{M}'
 \end{array}$$

$\tilde{\theta}'$ is injective $\implies \psi = 0$

$\mathbb{F}G \longrightarrow \mathbb{1}_{\text{Tri}(A)}$ is natural: exercise.



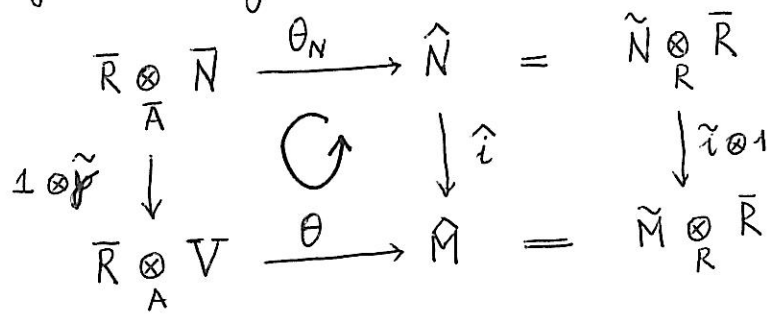
$$\begin{array}{ccc}
 \text{Hom}_A(N, \tilde{M}) & \xrightarrow{\sim} & \text{Hom}_R(\tilde{N}, \tilde{M}) \\
 \downarrow i & & \downarrow \tilde{\gamma} \\
 & \xrightarrow{\quad} &
 \end{array}$$

Since $Q(N) \xrightarrow[Q(i)]{\sim} Q(\tilde{M}) \Rightarrow Q(\tilde{N}) \xrightarrow[Q(\tilde{\gamma})]{\sim} Q(\tilde{M})$

[well, at this place we need $\Rightarrow \tilde{\gamma}$ is a monomorphism

A has no emb. primes $\Rightarrow \bar{A}, \bar{R}$ have smaller kr. dim than A

Diagram chasing:



$\Rightarrow \hat{i}$ is an epimorphism (θ and $1 \otimes \varphi$ are)

But $I \subset m = \text{rad}(A) \subset \text{rad}(R)$

Nakayama's Lemma: $\tilde{\gamma}$ is surjective \Rightarrow bijective

$\Rightarrow \mathbb{1}N \xrightarrow{\sim} \mathbb{1}\tilde{M}$

$\Rightarrow \mathbb{1}N \xrightarrow{\sim} \mathbb{1}\tilde{M} \Rightarrow \varphi$ is isom. $\Rightarrow \tilde{\gamma}$ is an isom.

• G is faithful

$$\begin{array}{ccccccc}
 0 & \longrightarrow & N & \longrightarrow & \tilde{M} \oplus V & \xrightarrow{(\pi, -\theta)} & \hat{M} \longrightarrow 0 \\
 & & \downarrow \rho & & \downarrow \begin{pmatrix} \rho & 0 \\ 0 & \gamma \end{pmatrix} & & \downarrow \varphi \\
 0 & \longrightarrow & N' & \longrightarrow & \tilde{M}' \oplus V' & \xrightarrow{(\pi', -\theta')} & \hat{M}' \longrightarrow 0
 \end{array}$$

$$P = G(\varphi, \psi)$$

$$\rho = 0 \iff \text{Im}(\rho) = 0$$

$$\begin{array}{ccc}
 Q(N) & \xrightarrow{\sim} & Q(\tilde{M}) \\
 Q(\rho) \downarrow & & \downarrow Q(\varphi) \\
 Q(N') & \xrightarrow{\sim} & Q(\tilde{M}')
 \end{array}$$

$$\Rightarrow Q(\varphi) = 0 \Rightarrow \text{Im}(\varphi) = 0 \Rightarrow \psi = 0$$

• Corollary: \mathbb{F} is full

$$\mathbb{F}(M) \xrightarrow{f} \mathbb{F}(M') \Rightarrow$$

$$\begin{array}{ccc}
 & & \mathbb{G}\mathbb{F}(g) \\
 & \curvearrowright & \\
 \mathbb{G}\mathbb{F}(M) & \xrightarrow{\mathbb{G}(f)} & \mathbb{G}\mathbb{F}(M') \\
 \cong \uparrow & & \uparrow \cong \\
 M & \xrightarrow{g} & M'
 \end{array}$$

$$\mathbb{G}\mathbb{F}(g) = \mathbb{G}(f) \xrightarrow{G \text{ is faithful}} f = \mathbb{F}(g)$$

• Remains to show: \mathbb{F} is dense

$$T = (\tilde{M}, V, \theta) \quad N = G(T)$$

$$\begin{array}{ccccccc}
 0 & \longrightarrow & I\tilde{M} & \longrightarrow & N & \xrightarrow{\gamma} & V \longrightarrow 0 \\
 & & \parallel & & \downarrow i & & \downarrow \theta \\
 0 & \longrightarrow & I\tilde{M} & \longrightarrow & \tilde{M} & \xrightarrow{\pi} & \hat{M} \longrightarrow 0
 \end{array}$$

$$\begin{array}{ccc}
 R \otimes_A N & \longrightarrow & \tilde{M} \\
 \downarrow & & \uparrow \tilde{i} \\
 \frac{R \otimes N}{\text{tor}} & &
 \end{array}
 \quad
 \begin{array}{ccc}
 \bar{A} \otimes_A N & \xrightarrow{\tilde{\gamma}} & V
 \end{array}$$

Claim

$$\mathbb{F}G(T) = (\tilde{N}, \hat{N}, \theta_N) \xrightarrow[\sim]{(\tilde{i} \ \tilde{\gamma})} (\tilde{M}, V, \theta) = T.$$

▣

Tutorium: CM modules over $k[x, y, z] / (xy, xz, yz)$.