Cohen-Macaulay modules over curve singularities

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§ I Preliminaries from commutative algebra

① A reduced Noetherian ring [no embedded associated primes is actually sufficient]

$\{\mathfrak{p}_1, \dots, \mathfrak{p}_n\} = \{\mathfrak{p} \in \operatorname{Spec}(A) \mid \operatorname{ht}(\mathfrak{p}) = 0\}$

$Q = Q(A)$ total ring of fractions

<u>Proposition</u>:
$$\begin{tikzcd} Q \arrow[r, "\sim"] & A_{\mathfrak{p}_1} \times \dots \times A_{\mathfrak{p}_n} = Q_1 \times \dots \times Q_n \\ & A \arrow[ul, bend left] \arrow[u, "can"'] \end{tikzcd}$$
fields; semi-simple

② Let $A \overset{\text{finite}}{\subset} R \subset Q$

<u>Def</u> $I := \operatorname{Ann}_A(R/A)$ conductor ideal

<u>Lemma</u> $I = \{a \in A \mid aR \subseteq A\}$

$I = IA = IR$

<u>Lemma</u>
$$\begin{array}{ccc} I & \xrightarrow{\sim} & \operatorname{Hom}_A(R, A) \\ \uplus & & \uplus \\ a & \longmapsto & (r \longmapsto ar) \end{array}$$

<u>Proof</u>
$$\begin{tikzcd} R \arrow[r, hook] \arrow[d, "\varphi"'] & Q \otimes_A R \arrow[r, "mult"] \arrow[d, "1 \otimes \varphi"] & Q \arrow[d, "mult_q"] \\ A \arrow[r, hook] & Q \otimes_A A \arrow[r] & Q \end{tikzcd} \implies q \in I \subset A, \quad q = \varphi(1_R)$$

<u>Corollary</u> $\forall \mathfrak{p} \in \operatorname{Spec}(A)$ $\quad I_{\mathfrak{p}} \xrightarrow[\sim]{can} \operatorname{Ann}_{A_{\mathfrak{p}}}(R_{\mathfrak{p}}/A_{\mathfrak{p}})$

<u>Proof</u>
$$\begin{tikzcd} I_{\mathfrak{p}} \arrow[r, "\sim"] \arrow[dr, dashed, "can"'] & \operatorname{Hom}_A(R, A)_{\mathfrak{p}} \arrow[d, "\cong / can"] \\ & \operatorname{Hom}_{A_{\mathfrak{p}}}(R_{\mathfrak{p}}, A_{\mathfrak{p}}) \end{tikzcd}$$

<u>Corollary</u> $A/I =: \overline{A} \hookrightarrow \overline{R} := R/I$

Then $\overline{A}$ and $\overline{R}$ are supported at those $\mathfrak{p}$: $A_{\mathfrak{p}} \neq R_{\mathfrak{p}}$.

③ Let $M \in A\text{-mod}$

rational envelope $Q_A(M) := Q \underset{A}{\otimes} M \cong Q_1^{p_1} \oplus \ldots \oplus Q_n^{p_n}$

$\underline{rk}(M) := (p_1, \ldots, p_n)$ multi-rank of M

<u>Def</u>: $\operatorname{tor}_A(M) := \ker(M \longrightarrow Q \underset{A}{\otimes} M)$ torsion part of M

$$\tilde{M} := R \otimes_A M / \operatorname{tor}_R(R \underset{A}{\otimes} M)$$

<u>Proposition</u>

(i) $\forall N \in R\text{-mod}$ we have: $\operatorname{tor}_R(N) = \operatorname{tor}_A(N)$ as A-modules

(ii) $\forall M \in TF(A)$ $\quad M \longrightarrow \tilde{M}$ is injective

$$IM \longrightarrow I\tilde{M} \text{ is isom.}$$

<u>Proof</u>

(i)
$$\begin{array}{ccc} Q \underset{A}{\otimes} N & \xrightarrow{\sim} & Q \underset{R}{\otimes} N = Q(N) \\ \uparrow & & \uparrow \\ N & = & N \end{array} \quad \Rightarrow \operatorname{tor}_A(N) = \operatorname{tor}_R(N)$$

(ii)
$$M \longrightarrow R \underset{A}{\otimes} M \longrightarrow R \underset{A}{\otimes} M / \operatorname{tor}(-) \quad \text{induces}$$
$$R \underset{A}{\otimes} M / \operatorname{tor}(-) = \tilde{M}$$

$$Q(M) \xrightarrow{\cong} Q(\tilde{M})$$

Let $K = \ker(M \longrightarrow \tilde{M})$. Then

- $K \in TF(A)$
- $\underline{rk}(K) = 0$

$$\Longrightarrow K \subseteq Q(K) = 0 \Longrightarrow K = 0$$

Next $\quad IM \longrightarrow I\tilde{M}$ is injective

surjectivity: $\quad I\tilde{M} \ni a[m \otimes b] = [m \otimes ab] = ab[m \otimes 1]$, $\quad a \in I$, $m \in M$, $b \in R$

(II) Main Construction

$(A, \mathfrak{m})$ CM curve singularity [e.g. reduced]

$$A \underset{\text{finite}}{\subset} R \subset Q$$

$$I = \mathrm{Ann}_A(R/A)$$

Observe: $\bar{A} = A/I$ and $\bar{R} = R/I$ are artinian, $\mathrm{TF}(A) = \mathrm{CM}(A)$

$$M \longmapsto \bar{M} = M/IM$$

$$\begin{tikzcd}
 & \mathrm{TF}(A) \arrow[dl, "R\otimes_A -"'] \arrow[r, "\bar{A}\otimes_A -"] \arrow[d] & \bar{A}\text{-mod} \arrow[d, "\bar{R}\otimes_{\bar{A}} -"] \arrow[dl, Rightarrow, "\theta"'] \\
R\text{-mod} \arrow[r, "-/\mathrm{tor}(-)"'] & \mathrm{TF}(R) \arrow[r, "\bar{R}\otimes_R -"'] & \bar{R}\text{-mod}
\end{tikzcd}$$

$$M \longmapsto \tilde{M}$$

A is reduced [or weaker: no embedded associated primes]

$$\theta_M : \bar{R} \underset{\bar{A}}{\otimes} \bar{A} \underset{A}{\otimes} M \xrightarrow{\sim} \bar{R} \underset{R}{\otimes} R \underset{A}{\otimes} M \twoheadrightarrow \bar{R} \otimes \underbrace{\left(\frac{R \otimes_A M}{\mathrm{tor}(-)}\right)}_{\tilde{M}}$$

Idea: recover M from the triple $(\tilde{M}, \bar{M}, \theta_M)$

Note:

$$\begin{tikzcd}
0 \arrow[r] & IM \arrow[r] \arrow[d, equal] & M \arrow[r] \arrow[d, hook] & \bar{A}\otimes_A M \arrow[r] \arrow[d, "\tilde{\theta}_M"] & 0 \\
0 \arrow[r] & I\tilde{M} \arrow[r] & \tilde{M} \arrow[r] & \bar{R}\otimes_R \tilde{M} \arrow[r] & 0
\end{tikzcd}$$

Snake Lemma: $\tilde{\theta}_M$ is injective.

- first version: Drozd-Roiter 1967
- current exposition [BD, "CM on non-isol. surf. sing", Appendix]

Def: category of triples $\mathrm{Tri}(A)$

- objects: triples $(\tilde{M}, V, \theta)$

$\tilde{M} \in \mathrm{TF}(R)$

$V \in \bar{A}\text{-mod}$

$$\begin{tikzcd}
\bar{R}\otimes_A V \arrow[rr, two heads, "\theta"] & & \bar{R}\otimes_R \tilde{M} \\
 & V \arrow[ul, hook, "\mathrm{can}"] \arrow[ur, "\tilde{\theta}"'] &
\end{tikzcd}$$

gluing map (in $\bar{R}$-mod)

$\tilde{\theta}$ is injective

- *morphisms*

$$(\tilde M, V, \theta) \xrightarrow{(\varphi,\psi)} (\tilde M', V', \theta')$$

$$\begin{tikzcd}
V \arrow[d, "\psi"'] & \bar R \otimes_A V \arrow[r, "\theta"] \arrow[d, "1\otimes\psi"'] \arrow[dr, phantom, "\circlearrowright"] & \bar R \otimes_R \tilde M \arrow[d, "1\otimes\varphi"] & \tilde M \arrow[d, "\varphi"] \\
V' & \bar R \otimes_A V' \arrow[r, "\theta'"] & \bar R \otimes_R \tilde M' & \tilde M'
\end{tikzcd}$$

<u>Theorem</u>:
$$\begin{array}{ccc} TF(A) & \xrightarrow{\mathbb{F}} & Tri(A) \\ \Psi & & \Psi \\ M & \longmapsto & (\tilde M, \; \bar M, \; \theta_M) \end{array}$$
is an equivalence of categories.

$\tilde M = R \otimes_A M/_{\text{tors}}$, $\quad \bar M = \bar A \otimes_A M$

<u>Proof</u>: quasi-inverse equivalence

$$\begin{array}{ccc} \mathbb{G} : Tri(A) & \longrightarrow & TF(A) \\ \Psi & & \\ (\tilde M, V, \theta) & \longmapsto & \cancel{M} \end{array}$$

$$\begin{tikzcd}
0 \arrow[r] & I\tilde M \arrow[r] \arrow[d, equal] & M \arrow[r, "\gamma"] \arrow[d, "i"] \arrow[dr, phantom, "\text{pull-back}"] & V \arrow[r] \arrow[d, "\tilde\theta"] & 0 \\
0 \arrow[r] & I\tilde M \arrow[r] & \tilde M \arrow[r, "\pi"] & \bar R \otimes_R \tilde M \arrow[r] & 0
\end{tikzcd}$$

$\bar R \otimes_R \tilde M = \hat M$

- $\tilde\theta$ is monom $\overset{\text{snake Lemma}}{\Longrightarrow}$ i is monom $\Rightarrow$ M is torsion free
- functoriality of pull-backs $\Rightarrow$ $\mathbb{G}$ is a functor $\quad 0 \to N \to \tilde M \oplus V \xrightarrow{(\pi,-\tilde\theta)} \hat M \to 0$
- $\forall\, M \in TF(A)$

$$\begin{tikzcd}
0 \arrow[r] & IM \arrow[r] \arrow[d, equal] & M \arrow[r] \arrow[d] & \bar A \otimes_A M \arrow[r] \arrow[d, "\tilde\theta_M"] & 0 \\
0 \arrow[r] & I\tilde M \arrow[r] & \tilde M \arrow[r] & \bar R \otimes_R \tilde M \arrow[r] & 0
\end{tikzcd}$$

$\Rightarrow \quad \mathbb{GF}(M) \xrightarrow{\sim} M \qquad \mathbb{GF} \longrightarrow \mathbb{1}_{TF(A)}$.

$\Rightarrow \quad \mathbb{F}$ is faithful.

<u>Lemma</u> Let $(\tilde M, V, \theta) \xrightarrow{(\varphi,\psi)} (\tilde M', V', \theta')$ be a morphism in $Tri(A)$.

Then $\varphi = 0 \Rightarrow \psi = 0$

<u>Proof</u>

$$\begin{tikzcd}
V \arrow[r] \arrow[d, "\psi"'] & \bar R \otimes_{\bar A} V \arrow[r, "\theta"] \arrow[d, "1\otimes\psi"'] & \bar R \otimes_R \tilde M \arrow[d, "1\otimes\varphi \,=\, 0"] \\
V' \arrow[r] \arrow[rr, bend right] & \bar R \otimes_{\bar A} V' \arrow[r, "\theta'"] & \bar R \otimes_R \tilde M'
\end{tikzcd}$$

$\tilde\theta'$ is injective $\Rightarrow$ $\psi = 0$

$\mathbb{F}\mathbb{G} \longrightarrow \mathbb{1}_{Tri(A)}$ is natural : exercise.

$$\begin{tikzcd}
0 \arrow[r] & IN \arrow[r] & \tilde{N} \arrow[r] \arrow[ddd, dashed, bend left, "\tilde{i}"] & \hat{N} \arrow[r] & 0 \\
0 \arrow[r] & IN \arrow[r] \arrow[u, equal] \arrow[d] & N \arrow[r] \arrow[u] \arrow[d, equal] & \bar{N} \arrow[r] \arrow[u, "\tilde{\theta}_N"'] \arrow[d, "\tilde{\varphi}"] & 0 \\
0 \arrow[r] & I\tilde{M} \arrow[r] \arrow[d, equal] & N \arrow[r, "\gamma"] \arrow[d, "i"'] & V \arrow[r] \arrow[d, "\tilde{\theta}"] & 0 \\
0 \arrow[r] & I\tilde{M} \arrow[r] & \tilde{M} \arrow[r] & \hat{M} \arrow[r] & 0
\end{tikzcd}$$

$\Longrightarrow \varphi$ is epim

$$\begin{tikzcd}
\mathrm{Hom}_A(N, \tilde{M}) \arrow[r, "\sim"] & \mathrm{Hom}_R(\tilde{N}, \tilde{M}) \\
i \arrow[r, mapsto] \arrow[u, phantom, "\in"] & \tilde{i} \arrow[u, phantom, "\in"]
\end{tikzcd}$$

Sinse $Q(N) \xrightarrow[Q(i)]{\sim} Q(\tilde{M}) \Longrightarrow Q(\tilde{N}) \xrightarrow[\sim]{Q(\tilde{i})} Q(\tilde{M})$

[well, at this place we need $\Longrightarrow \tilde{i}$ is a monomorphism
A has no emb. primes $\Rightarrow$
$\bar{A}, \bar{R}$ have smaller kr.dim than A]

Diagram chasing:

$$\begin{tikzcd}
\bar{R} \otimes_{\bar{A}} \bar{N} \arrow[r, "\theta_N"] \arrow[d, "1 \otimes \tilde{\varphi}"'] \arrow[dr, phantom, "\circlearrowright"] & \hat{N} \arrow[d, "\hat{i}"] & = & \tilde{N} \otimes_R \bar{R} \arrow[d, "\tilde{i} \otimes 1"] \\
\bar{R} \otimes_A V \arrow[r, "\theta"] & \hat{M} & = & \tilde{M} \otimes_R \bar{R}
\end{tikzcd}$$

$\Rightarrow \hat{i}$ is an epimorphism (θ and $1 \otimes \varphi$ are)

But $I \subset m = rad(A) \subset rad(R)$

Nakayama's Lemma: $\tilde{i}$ is surjective $\Longrightarrow$ bijective

$\Rightarrow IN \xrightarrow{\sim} I\tilde{M}$

$\Rightarrow IN \xrightarrow{\sim} I\tilde{M} \Longrightarrow \varphi$ is isom. $\Rightarrow \tilde{\gamma}$ is an isom.

- $\mathbb{G}$ is faithful

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] \arrow[d, "\rho"'] & \widetilde{M} \oplus V \arrow[r, "{(\pi, -\theta)}"] \arrow[d, "\begin{pmatrix} \varphi & 0 \\ 0 & \psi \end{pmatrix}"'] & \widehat{M} \arrow[r] \arrow[d, "\hat{\varphi}"] & 0 \\
0 \arrow[r] & N' \arrow[r] & \widetilde{M}' \oplus V' \arrow[r, "{(\pi', -\theta')}"] & \widehat{M}' \arrow[r] & 0
\end{tikzcd}$$

$$\boxed{\rho = \mathbb{G}(\varphi, \psi)}$$

$\rho = 0 \iff \mathrm{Im}(\rho) = 0$

$$\begin{tikzcd}
Q(N) \arrow[r, "\sim"] \arrow[d, "Q(\rho)"'] & Q(\widetilde{M}) \arrow[d, "Q(\varphi)"] \\
Q(N') \arrow[r, "\sim"] & Q(\widetilde{M})
\end{tikzcd}$$

$\Rightarrow Q(\varphi) = 0 \Longrightarrow \mathrm{Im}(\varphi) = 0 \Longrightarrow \psi = 0$

- Corollary: $\mathbb{F}$ is full

$\mathbb{F}(M) \xrightarrow{f} \mathbb{F}(M') \Rightarrow$

$$\begin{tikzcd}
\mathbb{G}\mathbb{F}(M) \arrow[r, "\mathbb{G}(f)"] \arrow[r, dashed, bend left, "\mathbb{G}\mathbb{F}(g)"] & \mathbb{G}\mathbb{F}(M') \\
M \arrow[u, "\cong"] \arrow[r, dashed, "g"] & M' \arrow[u, "\cong"']
\end{tikzcd}$$

$$\mathbb{G}\mathbb{F}(g) = \mathbb{G}(f) \underset{G \text{ is faithful}}{\Longrightarrow} f = \mathbb{F}(g)$$

- Remains to show: $\mathbb{F}$ is dense

$T = (\widetilde{M}, V, \theta)$ $\qquad N = \mathbb{G}(T)$

$$\begin{tikzcd}
0 \arrow[r] & I\widetilde{M} \arrow[r] \arrow[d, equal] & N \arrow[r, "\gamma"] \arrow[d, "i"'] & V \arrow[r] \arrow[d, "\theta"] & 0 \\
0 \arrow[r] & I\widetilde{M} \arrow[r] & \widetilde{M} \arrow[r, "\pi"] & \widehat{M} \arrow[r] & 0
\end{tikzcd}$$

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$$\begin{tikzcd}
R \otimes_A N \arrow[r] \arrow[d] & \widetilde{M} \\
\dfrac{R \otimes_A N}{\mathrm{tor}} \arrow[ru, dashed, "\tilde{i}"'] &
\end{tikzcd}
\qquad
\bar{A} \otimes_A N \xrightarrow{\tilde{\gamma}} V$$

Claim

$$\mathbb{F}\mathbb{G}(T) = (\widetilde{N}, \widehat{N}, \theta_N) \xrightarrow[\sim]{(\tilde{i} \ \tilde{\gamma})} (\widetilde{M}, V, \theta) = T.$$

$\blacksquare$

Tutorium: CM modules over $k[[x,y,z]]/(xy, xz, yz)$.