Vector bundles on curves of genus one – I

I

Let k be a field

<u>Def</u>: an elliptic curve over k is a <u>smooth</u> projective curve E

- $H^1(\mathcal{O}_E) \cong k$ (arithmetic genus is 1)
- $E(k) \neq \emptyset$ ($\exists$ a point over k on E)

<u>Theorem</u>

- $E \cong V(f) \subseteq \mathbb{P}^2_k$ where $f \in k[x,y,z]_3$
- $\mathrm{char}(k) \neq 2,3$ ~~or 0~~

  wlog $f = zy^2 - x^3 - pxz^2 - qz^3$

$k = \mathbb{C}$

Elliptic curve = complex torus $E_\tau = \mathbb{C}/\langle 1,\tau\rangle$ $\quad \mathrm{Im}(\tau) > 0$

$$\begin{array}{c} E_\tau = \mathbb{C}/\langle 1,\tau\rangle \\ \wr \big\downarrow \exp(2\pi i -) \\ E_q = \mathbb{C}^*/q^{\mathbb{Z}} \end{array} \qquad q = e^{2\pi i \tau}$$

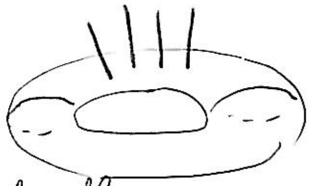
$$\mathbb{C}/\langle 1,\tau\rangle \xrightarrow{\text{"}(\wp(z) : \wp'(z) : 1)\text{"}} \mathbb{P}^2_{\mathbb{C}}$$

$$\wp(z) = \frac{1}{z^2} + \sum_{(m,n)\neq(0,0)} \left\{ \frac{1}{(z+m\tau+n)^2} - \frac{1}{(m\tau+n)^2} \right\}$$

<u>Theorem</u> (Weierstraß) $\quad \wp'^2 = 4\wp^3 - g_2\wp - g_3 \qquad g_2, g_3 \in \mathbb{C}$

Let $\varphi : \mathbb{C}^* \longrightarrow GL_n(\mathbb{C})$ be holomorphic

$$\begin{tikzcd} \mathbb{C}^* \times \mathbb{C}^n \arrow[r] \arrow[d] & \mathbb{C}^* \times \mathbb{C}^n /_{\sim} = \mathcal{E}(\varphi) \arrow[d, "p"] & \overline{(z,v)} \arrow[d, mapsto] \\ \mathbb{C}^* \arrow[r, "\pi"] & E_q \ni & \bar{z} \end{tikzcd} \qquad (z, v) \sim (qz, \varphi(z) v)$$



$\forall x \in E_q \quad p^{-1}(x) \cong \mathbb{C}^n$

$\Rightarrow \mathcal{E}(\varphi)$ is a holomorphic vector bundle of rank n on E_q

Theorem (Weil)

- $\forall \mathcal{E} \in VB(E_q) \quad \exists \varphi : \mathbb{C}^* \longrightarrow GL_n(\mathbb{C}) : \mathcal{E} \cong \mathcal{E}(\varphi)$
- $\mathcal{E}(\varphi) \cong \mathcal{E}(\psi) \iff \exists f \quad \dots \quad \psi(z) = f(qz)^{-1} \varphi(z) f(z)$

Theorem (Oda) Let $\mathcal{E} \in VB(E_q)$ be indecomposable.

Then $\exists!\, n, m, d \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0} \times \mathbb{Z}$, $\mathcal{L} \in \mathrm{Pic}^d(E_{q^n})$: $\boxed{\text{line bundle of degree } d}$,
$\gcd(n, d) = 1$

$$\mathcal{E} \cong \pi_*(\mathcal{L}) \otimes \mathcal{E}(N_m) \qquad N_m = \begin{pmatrix} 1 & 1 & & 0 \\ & 1 & \ddots & \\ & & \ddots & 1 \\ 0 & & & 1 \end{pmatrix}$$

$$\langle q^n \rangle \subseteq \langle q \rangle \subseteq \mathbb{C}^* \Rightarrow \underset{E_{q^n}}{\underset{\|}{\mathbb{C}^*/\langle q^n \rangle}} \xrightarrow{\pi} \underset{E_q}{\underset{\|}{\mathbb{C}^*/\langle q \rangle}} \quad \text{étale of degree } n$$

Rmk: C smooth projective curve $\quad p_1, \dots, p_t \in C$, $m_1, \dots, m_t \in \mathbb{Z}$ $\quad D = \sum_{i=1}^{t} m_i [p_i]$ Weil divisor

$\mathcal{L} := \mathcal{O}(D) \quad \deg(D) = \sum_{i=1}^{t} m_i$

Corollary $\mathcal{E} \cong \mathcal{E}(\Phi)$, where

$$\Phi = \begin{pmatrix} 0 & N_m & \dots & 0 \\ & & \ddots & \vdots \\ & & & N_m \\ \varphi^d N_m & 0 & \dots & 0 \end{pmatrix} \cdot \exp\left(-\frac{2\pi i x}{n}\right)$$

$\leftarrow n$ blocks

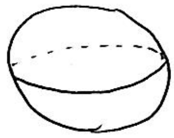
$\varphi = \exp(-2\pi i z - \pi i n \tau)$

function on $\mathbb{C}^*$!!!

Rmk This is the way how bundles on complex tori are used in mathematical physics

(II) Vector bundles on a projective line

<u>Remark</u> $k = \mathbb{C} \Rightarrow \mathbb{P}^1_{\mathbb{C}} \cong S^2$



<u>Lemma</u>

$$\begin{array}{ccc} \mathrm{Pic}(\mathbb{P}^1) & \xrightarrow[\deg]{\sim} & \mathbb{Z} \\ \cup & & \cup \\ \mathcal{O}_{\mathbb{P}^1}(n) & \longmapsto & n \\ \wr\wr & & \\ \mathcal{O}_{\mathbb{P}^1}(n[x]) & & \end{array}$$

$x \in \mathbb{P}^1$ any point

<u>Lemma</u>

$$\mathrm{Hom}_{\mathbb{P}^1}(\mathcal{O}(n), \mathcal{O}(m)) \cong \begin{cases} 0 & n > m \\ k[z_1, z_2]_{m-n} \cong k^{m-n+1} & \end{cases}$$

<u>Theorem</u> (Birkhoff, Grothendieck
Kronecker)

$$\forall\, \mathcal{F} \in VB(\mathbb{P}^1_k) \qquad \mathcal{F} \cong \bigoplus_{n \in \mathbb{Z}} \mathcal{O}_{\mathbb{P}^1}(n)^{\oplus m_n}$$

(III) Vector bundles on singular curves

Let C be a reduced curve over $k = \bar{k}$

- $\tilde{C} \xrightarrow{\pi} C$ normalization of C
- $\tilde{\mathcal{O}} := \pi_*(\mathcal{O}_{\tilde{C}}) \Rightarrow \forall x \in C \quad \tilde{\mathcal{O}}_x = \overline{\mathcal{O}}_{C,x}$ = integral closure of $\mathcal{O}_x$ in $Q(\mathcal{O}_x)$

$\mathcal{I} := \mathcal{A}nn_C(\tilde{\mathcal{O}}/\mathcal{O}) \xrightarrow{\sim} \mathcal{H}om_C(\tilde{\mathcal{O}}, \mathcal{O})$ conductor ideal sheaf

$\mathcal{A} = \mathcal{O}/\mathcal{I} \hookrightarrow \tilde{\mathcal{O}}/\mathcal{I} = \tilde{\mathcal{A}}$

$\Rightarrow \mathrm{Supp}(\mathcal{A}) = \mathrm{Supp}(\tilde{\mathcal{A}}) = \mathrm{Sing}(C)$

- Let $Z = V(\mathcal{I}) \cong \mathrm{Spec}(H^0(\mathcal{A}))$

$$\begin{tikzcd} \overset{\theta}{\tilde{Z}} \arrow[r, "\tilde{\eta}"] \arrow[d, "\tilde{\pi}"'] \arrow[dr, phantom, "\text{pull-back}"] & \overset{\tilde{\mathcal{F}}}{\tilde{C}} \arrow[d, "\pi"] \\ \underset{V}{Z} \arrow[r, hook, "\eta"] & C \end{tikzcd} \quad \leftarrow \text{dual to} \quad \begin{tikzcd} \tilde{\mathcal{A}} & \tilde{\mathcal{O}} \arrow[l] \\ \mathcal{A} \arrow[u] & \mathcal{O} \arrow[l] \arrow[u] \end{tikzcd}$$

$\tilde{Z} \cong \mathrm{Spec}(H^0(\tilde{\mathcal{A}}))$

Def: Category of triples $\mathrm{Tri}(C)$:

Objects: $(\tilde{\mathcal{F}}, V, \theta)$ $\quad \tilde{\mathcal{F}} \in VB(\tilde{C})$ $\quad \tilde{\pi}^* V \xrightarrow[\sim]{\theta} \tilde{\eta}^* \tilde{\mathcal{F}}$

$V \in VB(Z) \quad (\Rightarrow V \cong \mathcal{O}_Z^n)$

Morphisms: as in the case of modules

Theorem: The functor $VB(C) \xrightarrow{\mathbb{F}} \mathrm{Tri}(C)$ is an equivalence of categories

$$\mathcal{F} \longmapsto (\pi^*\mathcal{F}, \eta^*\mathcal{F}, \theta_{\mathcal{F}})$$

$$\theta_{\mathcal{F}} : \tilde{\pi}^* \eta^* \mathcal{F} \xrightarrow{\sim} \tilde{\eta}^* \pi^* \mathcal{F}$$

Idea of the proof

- quasi-inverse functor $\mathbb{G}: \mathrm{Tri}(C) \longrightarrow VB(C)$

$$T = (\tilde{\mathcal{F}}, V, \theta) \longmapsto \mathcal{F}$$

$$0 \longrightarrow \mathcal{F} \longrightarrow \pi_*(\widetilde{\mathcal{F}}) \oplus \eta_*(V) \xrightarrow{(can,\ \tilde\theta)} \nu_* \tilde\eta^*(\widetilde{\mathcal{F}}) \longrightarrow 0$$

where $\nu = \eta\tilde\pi = \pi\tilde\eta$

- $\widetilde{\mathcal{F}} \longrightarrow \tilde\eta_*\tilde\eta^*\mathcal{F} \implies \pi_*(\widetilde{\mathcal{F}}) \xrightarrow{can} \underbrace{\pi_*\tilde\eta_*}_{=\nu_*}\tilde\eta^*\widetilde{\mathcal{F}}$

- $\tilde\pi^* V \xrightarrow{\theta} \tilde\eta^*\widetilde{\mathcal{F}} \implies$

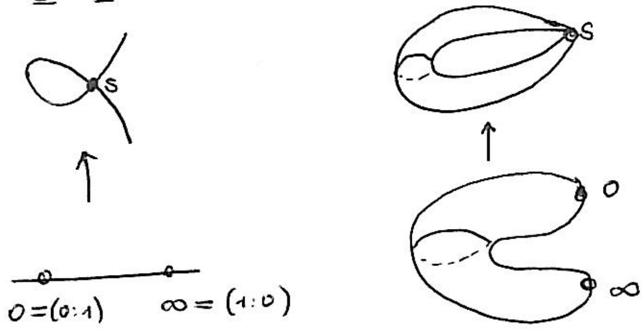
$$\begin{tikzcd}
\eta_*\tilde\pi_*\pi^*V \arrow[r, phantom, "="] & \nu_*\tilde\pi^*V \arrow[r, "\nu_*(\theta)"] & \nu_*\tilde\eta^*\widetilde{\mathcal{F}} \\
& \eta_*V \arrow[ul, "\eta_*(can)"] \arrow[ur, dashed, "\tilde\theta"'] &
\end{tikzcd}$$

- $\mathbb{1}_{VB(C)} \xrightarrow{\sim} \mathbb{G}\mathbb{F}$: follows from the local case
  $\mathbb{F}\mathbb{G} \xrightarrow{\sim} \mathbb{1}_{Tri(C)}$ $\square$

Main Example

- $E = V(zy^2 - x^3 - x^2z) \subset \mathbb{P}^2$

$$\begin{array}{c} E \\ \text{normalizat}\quad \pi \uparrow \\ \mathbb{P}^1 \end{array}$$



- $\widetilde{\mathcal{O}} = \pi_*(\mathcal{O}_{\mathbb{P}^1}) \qquad I = \mathrm{Ann}(\widetilde{\mathcal{O}}/\mathcal{O})$

$$\implies \begin{array}{l} A = \mathcal{O}/I \cong k_s \\ \downarrow \qquad\qquad\quad \searrow \text{diag} \\ \widetilde{A} = \widetilde{\mathcal{O}}/I \cong (k_0 \times k_\infty)_s \end{array}$$

$$\begin{tikzcd}
\widetilde{Z} \arrow[r, hook, "\tilde\eta"] \arrow[d, "\tilde\pi"'] & \mathbb{P}^1 \arrow[d, "\pi"] \\
Z \arrow[r, hook, "\eta"] & E
\end{tikzcd}$$

- Let $T = (\widetilde{\mathcal{F}}, V, \theta) \in Tri(E)$

$\implies \widetilde{\mathcal{F}} \in VB(\mathbb{P}^1) \implies \widetilde{\mathcal{F}} \cong \bigoplus_{c \in \mathbb{Z}} \mathcal{O}_{\mathbb{P}^1}(c)^{\oplus m_c}$

$V \in VB(Z) \implies V \cong k_s^n$

$$\begin{tikzcd}
\tilde{\pi}^* V \arrow[r, "\theta"] \arrow[d, "s"'] & \tilde{\eta}^* \tilde{\mathcal{F}} \arrow[d, "s"] \\
k_0^n \oplus k_\infty^n \arrow[r, "\begin{pmatrix} \theta_0 & 0 \\ 0 & \theta_\infty \end{pmatrix}"] & k_0^n \oplus k_\infty^n
\end{tikzcd}$$

choose certain trivializations of $\mathcal{O}_{\mathbb{P}^1}(c)$

$\Rightarrow \theta$ is given by $\theta_0, \theta_\infty \in GL_n(k)$.

- $(\tilde{\mathcal{F}}, V, \theta) \xrightarrow{\sim} (\tilde{\mathcal{F}}', V', \theta')$

  wlog $\tilde{\mathcal{F}} = \tilde{\mathcal{F}}'$

  $V = V'$

  $\exists\ \varphi \in \mathrm{Aut}(\tilde{\mathcal{F}}),\ \psi \in \mathrm{Aut}(V)$

$$\begin{tikzcd}
\tilde{\pi}^*(V) \arrow[r, "\theta"] \arrow[d, "\tilde{\pi}^*(\psi)"'] \arrow[dr, phantom, "\circlearrowright"] & \tilde{\eta}^* \tilde{\mathcal{F}} \arrow[d, "\tilde{\eta}^*(\varphi)"] \\
\tilde{\pi}^*(V) \arrow[r, "\theta'"] & \tilde{\eta}^* \tilde{\mathcal{F}}
\end{tikzcd}$$

- $V = k_s^n \quad \Rightarrow \quad \psi \in GL_n(k)$

- $\tilde{\mathcal{F}} = \bigoplus_{c \in \mathbb{Z}} \cancel{\mathcal{O}_{\mathbb{P}^1}(c)^{m_c}} \quad \mathcal{O}(c_1)^{m_1} \oplus \dots \oplus \mathcal{O}(c_t)^{m_t} \qquad c_1 < \dots < c_t$

$$\Rightarrow \cancel{\varphi \text{re}}\ \mathrm{Aut}(\tilde{\mathcal{F}}) \ni \varphi = \begin{pmatrix} \varphi_{11} & & & 0 \\ \varphi_{21} & \varphi_{22} & & \\ \vdots & \vdots & \ddots & \\ \varphi_{t1} & \varphi_{t2} & \cdots & \varphi_{tt} \end{pmatrix}$$

$$\varphi_{ij} \in \mathrm{Mat}_{m_i \times m_j}\left(k[z_1, z_2]_{c_i - c_j} \right)$$

$$\Rightarrow \quad \begin{tikzcd} k_0^n \oplus k_\infty^n \arrow[r, "\begin{pmatrix} \theta_0 & 0 \\ 0 & \theta_\infty \end{pmatrix}"] \arrow[d, "\begin{pmatrix} \psi & 0 \\ 0 & \psi \end{pmatrix}"'] & k_0^n \oplus k_\infty^n \arrow[d, "\begin{pmatrix} \varphi(0:1) & 0 \\ 0 & \varphi(1:0) \end{pmatrix}"] \\ k_0^n \oplus k_\infty^n \arrow[r, "\begin{pmatrix} \theta_0' & 0 \\ 0 & \theta_\infty' \end{pmatrix}"] & k_0^n \oplus k_\infty^n \end{tikzcd}$$

Remarq $\Rightarrow \begin{cases} \theta_0' = \varphi(0:1)\, \theta_0\, \psi^{-1} \\ \theta_\infty' = \varphi(1:0)\, \theta_\infty\, \psi^{-1} \end{cases}$

Remark : $\varphi(0:1) = \begin{pmatrix} \varphi_{11} & 0 & \cdots & 0 \\ \varphi_{11}^0 & \varphi_{12} & & 0 \\ \vdots & & \ddots & \vdots \\ \varphi_{t1}^0 & \cdots & & \varphi_{tt} \end{pmatrix}$

$\varphi(1:0) = \begin{pmatrix} \varphi_{11} & 0 & \cdots & 0 \\ \varphi_{21}^\infty & \varphi_{22} & & 0 \\ \vdots & & \ddots & \vdots \\ \varphi_{t1}^\infty & \cdots & & \varphi_{tt} \end{pmatrix}$

diagonal blocks are equal (but arbitrary)
under diagonal : any matrices