

## Vector bundles on curves of genus one - II

Theorem (Riemann-Roch formula) Let  $C$  be an integral curve over  $k = \bar{k}$   
 reduced + irreducible  
 and  $\mathcal{F} \in VB(C)$ . Then

$$\chi(\mathcal{F}) := \underset{\dim_k H^1(\mathcal{F})}{h^0(\mathcal{F})} - \underset{\deg(\det(\mathcal{F}))}{h^1(\mathcal{F})} = \underset{h^1(0)}{(1-g)\operatorname{rk}(\mathcal{F})}$$

Corollary Let  $E \subseteq \mathbb{P}_k^2$  be an integral cubic curve (e.g. )

Then  $\forall \mathcal{F} \in VB(E) \quad \chi(\mathcal{F}) = \deg(\mathcal{F})$ .

Lemma Let  $E = \begin{array}{c} s \\ \diagup \quad \diagdown \\ \text{---} \end{array} \xleftarrow{\pi} \begin{array}{c} | \\ \diagup \quad \diagdown \\ \mathcal{F} \end{array} = \mathbb{P}^1$ .

Then  $\forall \mathcal{F} \in VB(E) \quad \deg(\mathcal{F}) = \deg(\underset{\mathcal{F}}{\overset{s}{\mathcal{F}}} \mathcal{F})$ .

Proof :  $0 \longrightarrow \mathcal{F} \xrightarrow{\pi_* \mathcal{F}} \underset{\text{isomorphism on } E_{\text{reg}}}{\uparrow} \pi_* \mathcal{F} \longrightarrow \mathcal{J} \longrightarrow 0$        $\operatorname{Supp}(\mathcal{J}) = \{s\}$

- $\deg(\mathcal{F}) = \chi(\mathcal{F}) = \chi(\pi_* \mathcal{F}) - \chi(\mathcal{J})$        $[\pi \text{ is finite}]$   
 $= \chi(\mathcal{F}) - h^0(\mathcal{J})$

• Riemann-Roch on  $\mathbb{P}^1$

$$\chi(\mathcal{F}) = \deg(\mathcal{F}) + \underset{n}{\operatorname{rk}(\mathcal{F})}$$

•  $s \in U \subseteq E$

$$0 \longrightarrow \mathcal{F}|_U \xrightarrow{\text{SII}} \pi_* \mathcal{F}|_U \longrightarrow \mathcal{J}|_U \longrightarrow 0$$

$$\text{SII} \qquad \qquad \qquad \qquad \qquad \widetilde{\mathcal{O}} = \pi_*(\mathcal{O}_{\mathbb{P}^1})$$

$$\mathcal{O}_{U \cap}^n \longrightarrow \widetilde{\mathcal{O}}_{U \cap}^n$$

$$\Rightarrow \mathcal{J} \cong (\widetilde{\mathcal{O}}/\mathcal{O})^n$$

- $\widehat{\mathcal{O}}_{E,s} \cong k[[u,v]]_{uv} \Rightarrow \widetilde{\mathcal{O}}/\mathcal{O} \cong k_s \Rightarrow \deg(\mathcal{F}) = \deg(\mathcal{F})$

$$\widehat{\mathcal{O}}_{E,s} \cong k[[u]] \times k[[v]]$$



Last time

$$\text{Spec}(k \times k) = \tilde{\mathbb{Z}} \hookrightarrow \mathbb{P}^1$$

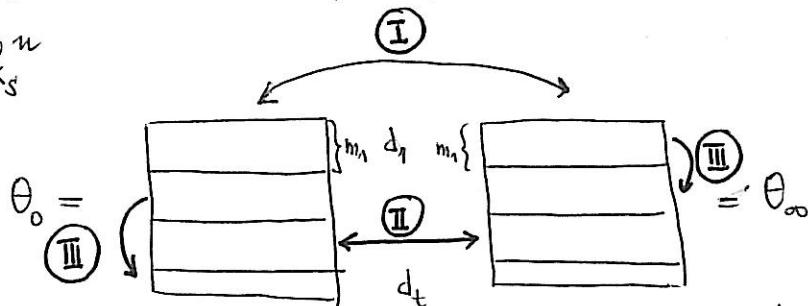
$$\downarrow \quad \int \pi \quad \rightsquigarrow \text{category } \text{Tri}(E)$$

$$\text{Spec}(k) \cong \mathbb{Z} \hookrightarrow E = \mathcal{X}$$

$$(\tilde{\mathcal{F}}, V, (\theta_0, \theta_\infty))$$

- $\tilde{\mathcal{F}} = \mathcal{O}_{\mathbb{P}^1}(d_1)^{m_1} \oplus \dots \oplus \mathcal{O}_{\mathbb{P}^1}(d_t)^{m_t} \quad d_1 < \dots < d_t$

- $V = k_s^n$



matrix problem  
representation of  
"bundle of chains  $\mathcal{X}$ "

$$\text{VB}(E) \xrightarrow[\sim]{\mathbb{F}} \text{Tri}(E) \xrightarrow[\sim]{\mathbb{H}} \text{Rep}(\mathcal{X})$$

$$(\tilde{\mathcal{F}}, V, (\theta_0, \theta_\infty)) \longmapsto (\theta_0, \theta_\infty)$$

Examples

$$\textcircled{1} \quad (\mathcal{O}_{\mathbb{P}^1}, k_s, (1), (1)) \longrightarrow \mathcal{O}_E$$

$$\textcircled{2} \quad k^* \xrightarrow[\sim]{} \text{Pic}^0(E) \quad \text{Jacobian}$$

$$\lambda \xrightarrow[\psi]{} (\mathcal{O}_{\mathbb{P}^1}, k_s, ((1), (\lambda))) \cong (\mathcal{O}_{\mathbb{P}^1}, k_s, ((\lambda^{-1}), (1)))$$

$$\textcircled{3} \quad k^* \times \mathbb{Z} \xrightarrow[\sim]{} \text{Pic}^\lambda(E)$$

$$(\lambda, d) \xrightarrow[\psi]{} (\mathcal{O}_{\mathbb{P}^1}(d), k_s, ((1), (\lambda)))$$

$$\textcircled{4} \quad H^1(\mathcal{O}_E) \cong \text{Ext}_E^1(\mathcal{O}_E, \mathcal{O}_E) \cong k$$

$$\Rightarrow \exists! \text{ non-split } 0 \longrightarrow \mathcal{O} \longrightarrow A \longrightarrow \mathcal{O} \longrightarrow 0$$

indecomposable [decomp  $\Rightarrow \oplus$  of lf  $\Rightarrow \mathbb{F}$ ]

Claim  $A \longrightarrow (\mathcal{O}_{\mathbb{P}^1}^2, k_s^2, ((1, 0), (0, 1)))$

More generally,  $\exists!$  non-split

$$0 \rightarrow A_{m-1} \rightarrow A_m \longrightarrow 0 \rightarrow 0 \quad A_1 = \mathcal{O}, A_2 = \mathcal{O}$$

indecomposable

Claim  $A_m \longrightarrow \left( \mathcal{O}_{\mathbb{P}^1}^m, k_s^m, \left( \begin{pmatrix} \lambda & \\ \vdots & 0 \end{pmatrix}, \begin{pmatrix} \lambda & \lambda & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & \lambda & \end{pmatrix} \right) \right)$

Theorem ( $\nearrow$  Bondarenko, 1988) Indecomposable objects of  $\text{Rep}(\mathcal{X})$  belonging to  $\text{Im}(\mathcal{H})$  have the following combinatorics

- $\mathbf{d} = (d_1, \dots, d_n) \in \mathbb{Z}^n$  non-periodic

- $m \in \mathbb{N}$
- $\lambda \in k^*$

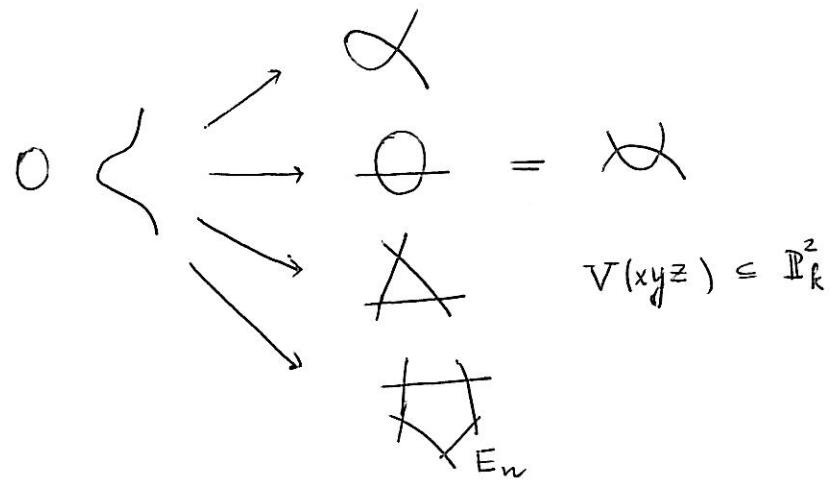
$$\Rightarrow \mathcal{I} = \begin{pmatrix} \lambda & & & 0 \\ \vdots & \ddots & & \\ 0 & \dots & 0 & \end{pmatrix} \quad \mathcal{J} = \begin{pmatrix} \lambda & & & 0 \\ \vdots & \ddots & & \\ 0 & \dots & 0 & \end{pmatrix}$$

Then  $\theta_0 = \begin{pmatrix} \mathcal{I} & 0 & \dots & 0 \\ 0 & \mathcal{I} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathcal{I} \end{pmatrix}$   $\theta_\infty = \begin{pmatrix} 0 & \mathcal{I} & \dots & 0 \\ 0 & 0 & \dots & \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \dots & \mathcal{I} \end{pmatrix} = \theta_\infty$

Example  $\mathbf{d} = (1, 0, 1)$

$$\begin{pmatrix} \mathcal{I} & 0 & 0 \\ 0 & \mathcal{I} & 0 \\ 0 & 0 & \mathcal{I} \end{pmatrix} \xrightarrow[1]{0} \begin{pmatrix} 0 & \mathcal{I} & 0 \\ 0 & 0 & \mathcal{I} \\ \mathcal{J} & 0 & 0 \end{pmatrix} \xrightarrow[1-2]{\sim} \begin{pmatrix} \mathcal{I} & 0 & 0 \\ 0 & \mathcal{I} & 0 \\ 0 & 0 & \mathcal{I} \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & \mathcal{I} \\ \mathcal{I} & 0 & 0 \\ 0 & \mathcal{J} & 0 \end{pmatrix}$$

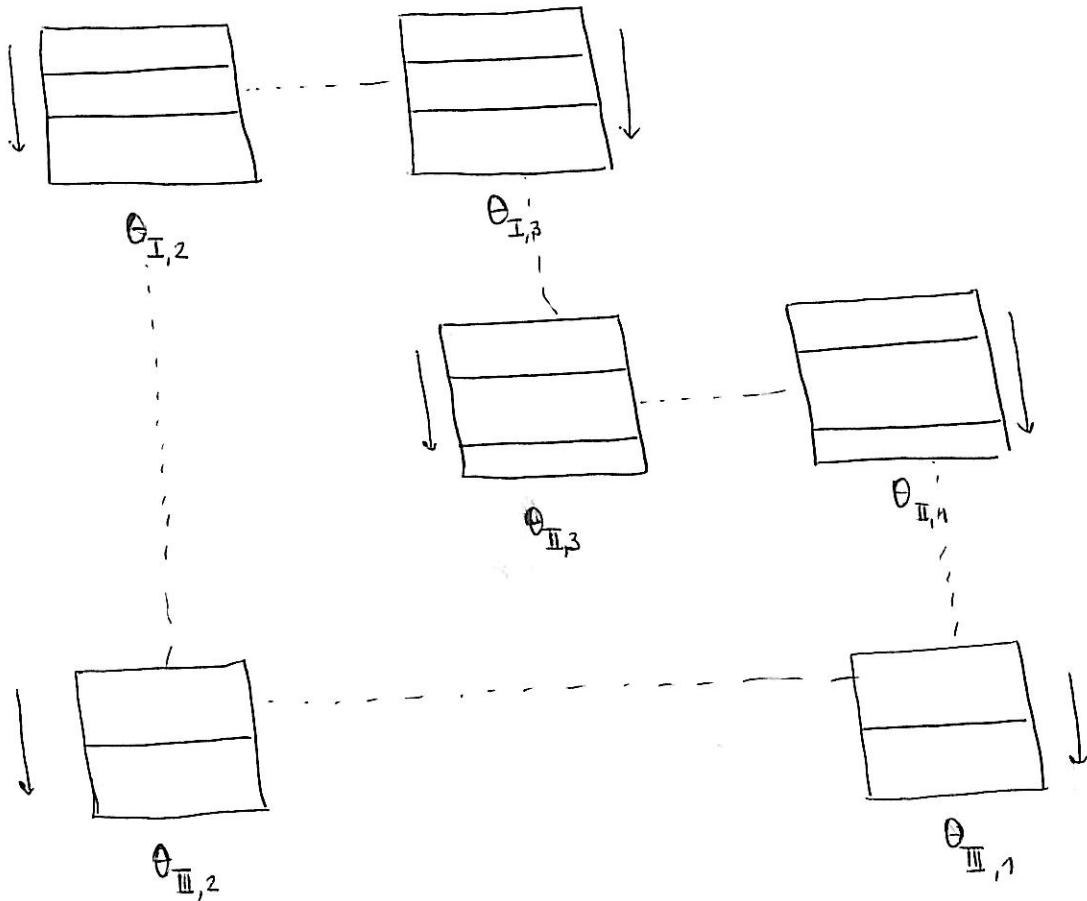
Remark



Matrix problem for  $\mathrm{Tr}(E_n)$  belongs to the same class ("representations of a bunch of chains")

Example

$$E = \begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \end{array}$$



Corollary : line bundles on  $E_n$

$n=3$

$$\begin{array}{c} [1] - d_1 - [1] \\ | \qquad | \\ [1] - d_2 - [1] \\ | \\ [1] - d_3 - \cdots - [1] \\ f^* \end{array}$$

$$\begin{array}{ccc} \text{Lemma} & \text{Pic}(E_n) & \xrightarrow{\sim} \mathbb{Z}^n \times k^* \\ & \mathcal{L} & \xrightarrow{\psi} (\deg(\mathcal{L}), \dots) \end{array}$$

$\deg(\mathcal{L}) = (d_1, \dots, d_n)$   
 $d_i = \deg(\mathcal{L}|_{E_n^{(i)}})$

Theorem (Drezd-Greuel, BB DG)

- Let  $E = E_1 = \bigcirc$  and  $\mathcal{F} \in \text{Ind}(\text{VB}(E))$ . Then  
 $\exists! n, m \in \mathbb{N} \quad \mathcal{L} \cong \mathcal{L}(dl, \lambda) \in \text{Pic}(E_n), \quad dl = (d_1, \dots, d_n) \text{ non-periodic}$   
 $\mathcal{F} \cong \mathcal{B}(dl, m, \lambda) := \pi_* (\mathcal{L}(dl, \lambda) \otimes A_m) \underset{\text{char}(k)=0}{\cong} \pi_* (\mathcal{L}(dl, \lambda)) \otimes A_m$

$$\begin{array}{ccc} \text{Diagram of } E_n & \xrightarrow{\pi} & \text{Diagram of } E_1 \\ \text{Etale covering of degree } n & & \end{array}$$

- similarly,  $\forall \mathcal{F} \in \text{VB}(E_p)$  indecomp.  
 $\exists! n, m \in \mathbb{N} \quad \text{Pic } \mathcal{L} \cong \mathcal{L}(dl, \lambda) \in \text{Pic}(E_{pn}), \quad dl \text{ non-periodic}$   
 $\mathcal{F} \cong \mathcal{B}(dl, m, \lambda) := \pi_* (\mathcal{L}(dl, \lambda) \otimes A_m)$

$$\begin{array}{ccc} \text{Diagram of } E_n & \xrightarrow{\pi} & \text{Diagram of } E_1 \\ \text{Etale covering of degree } n & & \end{array}$$

Question: how to compute cohomologies of  $\mathcal{B}(dl, m, \lambda)$ ?

- $H^i(E_p, \mathcal{B}(dl, m, \lambda)) \cong H^i(E_{pn}, \mathcal{L}(dl, \lambda) \otimes A_m) \quad i=0, 1$
- $\chi(\mathcal{L}(dl, \lambda) \otimes A_m) = m(d_1 + \dots + d_{pn})$

$\Rightarrow$  suffices to compute  $h^0$

$$\begin{array}{ccc} \text{Diagram of } E_n & \xleftarrow{\pi} & \text{Diagram of } \tilde{E} \\ \text{Etale covering of degree } n & & \end{array}$$

$$\mathcal{L} = \mathcal{L}(dl, \lambda) \Rightarrow \tilde{\mathcal{L}} := \pi^*(\mathcal{L}) \cong \bigoplus_{i=1}^n \mathcal{O}_{\tilde{E}_{(i)}}(d_i)$$

$$0 \rightarrow \mathcal{L} \longrightarrow \pi_x^* \pi^* \mathcal{L} \longrightarrow k_{s_1} \oplus \dots \oplus k_{s_n} \rightarrow 0$$

$$\Rightarrow 0 \rightarrow H^0(E, \mathcal{L}) \longrightarrow H^0(\tilde{E}, \tilde{\mathcal{L}}) \xrightarrow{\Psi} k^n \longrightarrow H^1(E, \mathcal{L}) \longrightarrow H^1(\tilde{E}, \tilde{\mathcal{L}}) \rightarrow 0$$

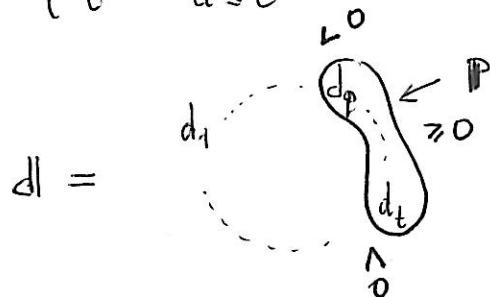
$$p_i = \begin{cases} 0 & \text{if } d_i < 0 \\ \in k[x_i, y]_{d_i} & \text{if } d_i \geq 0 \end{cases} \quad \rightsquigarrow \text{get explicit formula for } h^0 \text{ and } h^1$$

Corollary (Drezd-Grauel-Kashuba)

$$h^0(B(d, m, \lambda)) = m \left( \sum_{i=1}^{pn} (d_{i+1})^+ - \Theta(d) \right) + \delta(d, \lambda)$$

$$\delta(d, \lambda) = \begin{cases} 1 & d = (0, \dots, 0) \text{ and } \lambda = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$d^+ = \begin{cases} d & \text{if } d > 0 \\ 0 & \text{if } d \leq 0 \end{cases}$$



$$\Theta(p) = \begin{cases} t-q & p = (0, \dots, 0) \\ t-q+1 & \text{or } p = d \\ & \text{otherwise} \end{cases}$$

$$\Theta(d) = \sum_{\substack{p \subseteq d \\ \text{"positive"}}} \Theta(p)$$

"positive"

What about  $E = V(z^2 - x^3)$

Def.: let  $C$  be an integral projective curve

- for  $\mathcal{F} \in \text{coh}(C)$   $\mu(\mathcal{F}) := \frac{x(\mathcal{F})}{h(\mathcal{F})} \in \mathbb{Q} \cup \{\infty\}$
- $\mathcal{F}$  is called semi-stable if  $\forall 0 \rightarrow \mathcal{G} \rightarrow \mathcal{F}$  holds  $\mu(\mathcal{G}) \leq \mu(\mathcal{F})$ .
- Let  $\mathcal{S}^\mu(C) = \{\text{semi-stable sheaves of slope } \mu\}$

Lemma  $\mathcal{C}^\mu$  is abelian.

Theorem Let  $E = V(z^2 - 4x^3 - g_2xz^2 - g_3z^3)$

- $\forall \mathcal{F} \in \mathcal{C}^\mu$   $H^0(\mathcal{F}) \otimes \mathcal{O} \xrightarrow{\text{ev}_\mathcal{F}} \mathcal{F}$  is injective

- $\begin{cases} \mathcal{C}^\mu \longrightarrow \text{Tor}(E) \\ \mathcal{F} \longmapsto \text{cok}(\text{ev}_\mathcal{F}) \end{cases}$  is an equivalence of categories.

Remark  $\text{Tor}(E) \simeq \bigoplus_{x \in E} (\hat{\mathcal{O}}_{E,x} - \text{fdmod})$

Thm (Drozd, 1972)

$\mathcal{O} \in k[[u,v]]/(u^2-v^3) - \text{fdmod}$  is wild:

$\forall$  fin-dim alg.  $\wedge \exists$   $A\text{-mod} \xrightarrow[\text{exact, fully-faithful}]{} \mathcal{O}\text{-fdmod}$

Corollary  $\text{VB}(\langle \rangle)$  is representation-wild.

Theorem (Atiyah) Let  $E$  be an elliptic curve over  $k = \bar{k}$

$M_E(n,d) = \{\text{indecomp. vector bundles of rank } n\}$   
and degree  $d$  on  $E$

$$\gcd(n,d)=1 \quad M_E(nh, dh) \xleftarrow[\sim]{A_h \otimes -} M_E(n, d) \xrightarrow[\sim]{\det} \text{Pic}^d(E) \xrightarrow{\sim} E$$

$\text{Pic}^0(E)$  acts transitively

An indec. vb. on  $E$  is determined by its rank, degree and a point of  $E$