Letures 1–2 Drozd

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s1 ## 1. Cohen–Macaulay modules and rings

We denote by A a local (commutative) Noetherian ring with the maximal ideal $\mathfrak{m}$ and the residue field $\mathbf{k} = A/\mathfrak{m}$. For an A-module M we set $\operatorname{Kr.dim} M = \operatorname{Kr.dim} A/\operatorname{ann}_A M$.

11 **Definition 1.1.** Let M be an A-module. A sequence $\mathbf{a} = (a_1, a_2, \dots, a_m)$ of elements of $\mathfrak{m}$ is said to be an *M-sequence* if a_1 is a non-zero-divisor on M and for each i, $1 < i \leqslant n$, a_i is a non-zero-divisor on $M/(a_1, a_2, \dots, a_{i-1})M$.

12 **Theorem–Definition 1.2.** *Let M be a finitely generated A-module. The following conditions are equivalent:*

(1) $\operatorname{Ext}^i_A(\mathbf{k}, M) = 0$ *for all $i < d$.*
(2) $\operatorname{Ext}^i_A(N, M) = 0$ *for all $i < d$ and any A-module N of finite length.*
(3) $\operatorname{Ext}^i_A(N, M) = 0$ *for all $i < d$ and some A-module N of finite length.*
(4) *There is an M-sequence of length d.*

The biggest d with these properties is called the *depth* of M and denoted by $\operatorname{depth}_A M$.

Proof. Obviously, (1) $\Rightarrow$ (2) $\Rightarrow$ (3). Let N be an A-module of finite length, $d_N(M) = \min\{\, i \mid \operatorname{Ext}^i_A(N, M) \neq 0 \,\}$ and $d_s(M)$ be the maximal length of an M-sequence. If $d_s(M) = 0$, every element of $\mathfrak{m}$ is a zero divisor on M. Hence $\mathfrak{m} \in \operatorname{Ass}_A M$ and there is a submodule in M isomorphic to $A/\mathfrak{m} = \mathbf{k}$. Since N has a quotient module isomorphic to $\mathbf{k}$, $\operatorname{Hom}_A(N, M) \neq 0$, so $d_N(M) = 0$ too. If $d_s(M) = d > 0$, choose an M-sequence $(a = a_1, a_2, \dots, a_d)$. Then the sequence $0 \to M \xrightarrow{a} M \to M/aM \to 0$ is exact. It gives an exact sequence

$$\operatorname{Ext}^{i-1}_A(N, M) \to \operatorname{Ext}^{i-1}_A(N, M/aM) \to \operatorname{Ext}^i_A(N, M) \xrightarrow{a} \operatorname{Ext}^i_A(N, M).$$

If $i < d_N(M)$, it implies that $\operatorname{Ext}^{i-1}_A(N, M/aM) = 0$. If $i = d_N(M)$, $\operatorname{Ext}^i_A(N, M)$ is a nonzero module of finite length. Therefore, a is a zero divisor on this module, whence $\operatorname{Ext}^{i-1}_A(N, M/aM) = 0$. It means that $d_N(M/aM) = d_N(M) - 1$. Since $d_s(M/aM) = d - 1$, an induction on $d_s(M)$ gives that $d_s(M) = d_N(M)$, that is (3) $\Leftrightarrow$ (4). Moreover, since it is true for an arbitrary A-module of finite lenght, (1) $\Leftrightarrow$ (4) too. $\square$

Obviously, $\operatorname{depth}_A M = \operatorname{depth}_{A/\operatorname{ann}_A M} M$.

13 **Corollary 1.3.** *Let M be a finitely generated A-module, $d = \operatorname{depth}_A M$, and $\mathbf{a} = (a_1, a_2, \dots, a_m)$ be an M-sequence.*

(1) $\operatorname{depth}_A M/\mathbf{a}M = d - m$.
(2) *There is an M-sequence $(a_1, a_2, \dots, a_d)$ containing* $\mathbf{a}$.

Proof. The proof above implies that $\operatorname{depth}_A(M/aM) = \operatorname{depth}_A M - 1$ if a is a non-zero-divisor on M, so (1) follows by induction. Then, to obtain (2), one only has to choose an $M/\mathbf{a}M$-sequence $(a_{m+1}, \dots, a_d)$ of length $d - m$. $\square$

$\boxed{14}$ **Proposition 1.4.** *If $\mathfrak{p} \in \operatorname{Ass}_A M$, then $\operatorname{depth}_A M \leqslant \operatorname{Kr.dim} A/\mathfrak{p}$. In particular, $\operatorname{depth}_A M \leqslant \operatorname{Kr.dim} M$.*

Proof. We use the induction by $d = \operatorname{depth}_A M$, the case $d = 0$ being trivial. Let $d > 0$ and a is a non-zero-divisor on M, that is $a \notin \bigcup_{\mathfrak{p} \in \operatorname{Ass}_A M} \mathfrak{p}$. If $\mathfrak{p} \in \operatorname{Ass}_A M$, then $\operatorname{Hom}_A(A/\mathfrak{p}, M) \neq 0$. The exact sequence $0 \to M \xrightarrow{a} M \to M/aM \to 0$ induces the exact sequence

$$0 \to \operatorname{Hom}_A(A/\mathfrak{p}, M) \xrightarrow{a} \operatorname{Hom}_A(A/\mathfrak{p}, M) \to \operatorname{Hom}_A(A/\mathfrak{p}, M/aM),$$

whence $\operatorname{Hom}_A(A/\mathfrak{p}, M/aM) \neq 0$ by Nakayama's lemma. Therefore, $\mathfrak{p} + aA$ annihilate some nonzero element of M, hence $\mathfrak{p} + aA \subseteq \mathfrak{q}$ for some $\mathfrak{q} \in \operatorname{Ass}_A M/aM$. By the inductive supposition,

$$\begin{aligned}\operatorname{depth}_A M/aM = \operatorname{depth}_A M - 1 &\leqslant \operatorname{Kr.dim} A/\mathfrak{q} \leqslant \\ &\leqslant \operatorname{Kr.dim} A/(\mathfrak{p} + aA) \leqslant \operatorname{Kr.dim} A/\mathfrak{p} - 1.\end{aligned}$$ $\square$

$\boxed{15}$ **Proposition 1.5.** (1) *A sequence of elements $(a_1, a_2, \dots, a_m)$ from $\mathfrak{m}$ is an M-sequence if and only if $\operatorname{Ext}^m_A(A/\mathbf{a}, M) \neq 0$, while $\operatorname{Ext}^i_A(A/\mathbf{a}, M) = 0$ for $i < m$. In this case $\operatorname{Ext}^m_A(A/\mathbf{a}, M) \simeq M/\mathbf{a}M$.*

(2) *If $(a_1, a_2, \dots, a_m)$ is an M-sequence and all elements a_i belong to an ideal I, then $\operatorname{Ext}^i_A(N, M) = 0$ for $i < m$ and any A-module N such that $IN = 0$.*

Proof. We use induction on m. Let $a = a_1$, $\overline{M} = M/aM$. Obviously, a is a non-zero-divisor on M if and only if $\operatorname{Hom}_A(A/aA, M) = 0$. So we can suppose that a is indeed a non-zero-divisor on M. Consider the exact sequence $0 \to M \xrightarrow{a} M \to \overline{M} \to 0$. It gives the exact sequence

$$\begin{aligned}&\operatorname{Hom}_A(N, M) = 0 \to \operatorname{Hom}_A(N, \overline{M}) \to \operatorname{Ext}^1_A(N, M) \xrightarrow{0} \\ &\qquad \xrightarrow{0} \operatorname{Ext}^1_A(N, M) \to \operatorname{Ext}^1_A(N, \overline{M}) \to \operatorname{Ext}^2_A(N, M) \xrightarrow{0} \\ &\qquad \dots \xrightarrow{0} \operatorname{Ext}^i_A(N, M) \to \operatorname{Ext}^i_A(N, \overline{M}) \to \operatorname{Ext}^{i+1}_A(N, M) \xrightarrow{0} \dots\end{aligned}$$

since $a \operatorname{Ext}^i_A(N, M) = 0$. It implies both claims if $m = 1$. If $m > 1$, it implies that

$$\min\{\, i \mid \operatorname{Ext}^i_A(N, M) \neq 0 \,\} = \min\{\, i \mid \operatorname{Ext}^i_A(N, \overline{M}) \neq 0 \,\} + 1,$$

whence both claims follow by the inductive supposition applied to $\overline{M}$ and the sequence $(a_2, a_3, \dots, a_m)$. □

16 **Corollary 1.6.** *If $(a_1, a_2, \dots, a_m)$ is an M-sequence and σ is any permutation of indices, then $(a_{\sigma 1}, a_{\sigma 2}, \dots, a_{\sigma m})$ is also an M-sequence.*

17 **Definition 1.7.** An A-module M is said to be *Cohen–Macaulay module* if $\operatorname{depth}_A M = \operatorname{Kr.dim} M$. If, moreover, $\operatorname{Kr.dim} M = \operatorname{Kr.dim} A$, M is called *maximal Cohen–Macaulay.* If a ring A is a Cohen–Macaulay A-module, it is called a *Cohen–Macaulay ring.*

We denote by $\mathsf{CM}(A)$ the category of maximal Cohen–Macaulay A-modules.

18 **Example 1.8.**

(1) Every module or ring of Krull dimension 0 is obviously Cohen–Macaulay.

(2) If $\operatorname{Kr.dim} A = 1$ and A is *reduced*, i.e. has no nilpotent ideals, it is Cohen–Macaulay. In this case maximal Cohen–Macaulay A-modules coincide with *torsion free* A-modules M, i.e. such that $ax \neq 0$ for every nonzero $x \in M$ and everynon-zero-divisor $a \in A$.

(3) If $\operatorname{Kr.dim} A = 2$ and A is *normal*, i.e. integral and integrally closed in its field of fractions, it is Cohen–Macaulay and maximal Cohen–Macaulay A-modules coincide with *reflexive* A-modules M, i.e. such that the natural map $M \to M^{**}$ is an isomorphism, where $M^* = \operatorname{Hom}_A(M, A)$ (see Section 4).

(4) If A is *regular*, i.e. $\operatorname{gl.dim} A < \infty$, it is Cohen–Macaulay and maximal Cohen–Macaulay A-modules coincide with free (or, the same, projective) A-modules. (It follows from the Auslander–Buchsbaum formula, see below).

(5) A is said to be a (local) *complete intersection* if $A \simeq R/\mathbf{a}$, where R is regular and $\mathbf{a} = (a_1, a_2, \dots, a_m)$, where $m = \operatorname{Kr.dim} R - \operatorname{Kr.dim} A$. It is known that in this case $(a_1, a_2, \dots, a_m)$ is an A-sequence, so the preceeding example and Corollary 1.3 imply that A is Cohen–Macaulay.

Corollary 1.3 and Proposition 1.4 imply

19 **Corollary 1.9.** *Let M be a Cohen–Macaulay A-module of Krull dimension d, $\overline{A} = A/\operatorname{ann}_A M$.*

(1) *If $\mathfrak{p} \in \operatorname{Ass}_A M$, then $\operatorname{Kr.dim} A/\mathfrak{p} = d$, so $\mathfrak{p}$ is minimal among prime ideals containing $\operatorname{ann}_A M$.*

(2) *$\mathbf{a} = (a_1, a_2, \dots, a_m)$, where $a_i \in \mathfrak{m}$, is an M-sequence if and only if $\operatorname{Kr.dim} M/\mathbf{a}M = d - m$. In this case $M/\mathbf{a}M$ is also a Cohen–Macaulay A-module.*

(3) *If* $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_m = \mathfrak{m}$ *is a chain of prime ideals such that* $\mathfrak{p}_0 \in \operatorname{Ass}_A M$ *and, for every* $0 \leqslant i < m$, $\mathfrak{p}_{i+1}$ *is minimal among prime ideals properly containing* $\mathfrak{p}_i$, *then* $m = d$.

(4) $\operatorname{Ass}_A M$ *consists of the minimal ideals from* $\operatorname{supp}_A M$ *and, for any prime ideal* $\mathfrak{p}$ *from* $\operatorname{spec} \overline{A}$, $\operatorname{ht} \mathfrak{p} + \operatorname{Kr.dim} \overline{A}/\mathfrak{p} = d$.

(5) *If* M *is a maximal Cohen–Macaulay* A*-module, then any* A*-sequence is an* M*-sequence as well.*

(6) *If* A *is a Cohen–Macaulay ring of Krull dimension* d, *then* $\operatorname{ht} \mathfrak{p} + \operatorname{Kr.dim} A/\mathfrak{p} = d$ *for any* $\mathfrak{p} \in \operatorname{spec} A$ *and the set of zero divisors in* A *coincides with the union of minimal prime ideals.*

Proof. (1) follows from Proposition 1.4.

(2) An element $a \in A$ is a non-zero-divisor on M if and only if $a \notin \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Ass}_A M$. Thus (1) imlies that this condition is equivalent to the equality $\operatorname{Kr.dim} M/aM = d - 1$. Now (2) follows by an obvious induction on d.

(3) Use the induction on d, the case $d = 0$ being trivial. If $d > 0$, then $m > 0$ too. Since $\mathfrak{p}_1 \notin \operatorname{Ass}_A M$, there is an element $a \in \mathfrak{p}_1$ which is a non-zero-divisor on M. Hence M/aM is a Cohen–Macaulay module of Krull dimension $d - 1$ and $\mathfrak{p}_1 \in \operatorname{Ass}_A M/aM$. By the inductive supposition, $m - 1 = d - 1$ and $m = d$.

Now (4–6) are immediate. □

The following result follows immediately from the long exact sequence for the functors Ext^i_A.

10 **Lemma 1.10** (Depth lemma). *If* $0 \to L \to M \to N \to 0$ *is an exact sequence of* A*-modules, then*

- $\operatorname{depth}_A M \geqslant \min\{\operatorname{depth}_A L, \operatorname{depth}_A N\}$.
- $\operatorname{depth}_A L \geqslant \min\{\operatorname{depth}_A M, \operatorname{depth}_A N + 1\}$.
- $\operatorname{depth}_A N \geqslant \min\{\operatorname{depth}_A M, \operatorname{depth}_A L - 1\}$.

In particular, if $\operatorname{depth}_A L < \operatorname{depth}_A M$ *or* $\operatorname{depth}_A N < \operatorname{depth}_A M$, *then* $\operatorname{depth}_A L = \operatorname{depth}_A N + 1$.

1a **Corollary 1.11.** *If* $\operatorname{Kr.dim} A = n$ *and in an exact sequence* $0 \to M_n \to M_{n-1} \to \cdots \to M_1 \to M_0 \to L \to 0$ *the modules* $M_0, M_1, \ldots, M_{n-1}$ *are maximal Cohen–Macaulay, so is also the module* M_n. *In particular, it is the case if* A *is Cohen–Macaulay and* $M_0, M_1, \ldots, M_{n-1}$ *are projective (for instance, if* M_n *is the* n*-th syzygy of* L*).*

1b **Theorem 1.12** (Auslander–Buchsbaum formula). *Let* M *be a finitely generated* A*-module such that* $\operatorname{pr.dim}_A M < \infty$. *Then*

$$\operatorname{depth}_A M + \operatorname{pr.dim}_A M = \operatorname{depth}_A A.$$

In particular, if A is regular, then

$$\operatorname{depth}_A M + \operatorname{pr.dim}_A M = \operatorname{Kr.dim} A$$

(Recall that in this case $\operatorname{Kr.dim} A = \operatorname{gl.dim} A = \operatorname{depth}_A A$.)

The proof will use some facts about projective (that is free) resolutions, Recall that a projective resolution of a module M

e11 (1.1) $$\cdots \to P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \to 0$$

is said to be *minimal* if $\operatorname{Ker} d_i \subseteq \mathfrak{m}P_i$ for all i. If M is finitely generated, such resolution always exists and is unique up to an isomorphism (not canonically). Moreover, if (1.1) is a minimal projective resolution of M, then the induced maps

$$\operatorname{Hom}_A(P_{i-1}, M) \xrightarrow{d_i^*} \operatorname{Hom}_A(P_i, M) \quad \text{and} \quad \mathbf{k} \otimes_A P_i \xrightarrow{1 \otimes D_i} \mathbf{k}_A P_{i-1}$$

are zero, hence

$$\operatorname{rk} P_i = \dim_{\mathbf{k}} \operatorname{Ext}^i_A(\mathbf{k}, M) = \dim_{\mathbf{k}} \operatorname{Tor}^A_i(\mathbf{k}, M).$$

In particular,

$$\operatorname{pr.dim}_A M = \max\{\, i \mid P_i \neq 0 \,\}.$$

1c **Lemma 1.13.** *Suppose that*

$$0 \to M_n \to M_{n-1} \to \cdots \to M_1 \to M_0 \to 0$$

is an exact sequence and $a \in A$ is a non-zero-divisor on $M_0, M_1, \dots, M_{n-1}$. Then it is a non-zero-divisor on M_n too and the sequence

$$0 \to M_n/aM_n \to M_{n-1}/aM_{n-1} \to \cdots \to M_1/aM_1 \to M_0/aM_0 \to 0$$

is also exact.

Proof. It is an easy induction, starting from the case $n = 2$, when it is a consequence of 3×3 lemma. □

An obvious induction gives now the following result.

1d **Corollary 1.14.** *Let $fA = (a_1, a_2, \dots, a_m)$ be both A-sequence and M-sequence, and (1.1) be a minimal projective resolution of M. Then the induced sequence*

$$\cdots \to P_n/\mathbf{a}P_n \to P_{n-1}/\mathbf{a}P_{n-1} \to \cdots \to P_1/\mathbf{a}P_1 \to P_0/\mathbf{a}P_0 \to M/\mathbf{a}M \to 0$$

is a minimal projective resolution of the $A/\mathbf{a}$-module $M/\mathbf{a}M$. In particular,

$$\operatorname{pr.dim}_{A/\mathbf{a}} M/\mathbf{a}M = \operatorname{pr.dim}_A M.$$

Proof of Theorem 1.12. . We use the induction on $d = \operatorname{depth}_A A$. Suppose first that $d = 0$ and $n = \operatorname{pr.dim}_A M > 0$. and consider a minimal projective resolution (1.1). There is a nonzero homomopphism (hence, a monomorphism) $\alpha : \mathbf{k} \to A$. It gives a commutative diagram

$$\begin{tikzcd}
\mathbf{k} \otimes_A P_n \arrow[r, "1\otimes d_n"] \arrow[d, "\alpha\otimes 1"'] & \mathbf{k} \otimes_A P_{n-1} \arrow[d, "\alpha\otimes 1"] \\
P_n \arrow[r, "d_n"] & P_{n-1}
\end{tikzcd}$$

where we identify P_i with $A\otimes_A P_i$. Since P_i are projective, both vertical maps are monomorphisms. Since d_n is also a monomorphim, so is $1\otimes d_n$, which is impossible, since we know that $1 \otimes d_n = 0$, while $\mathbf{k} \otimes_A P_n \neq 0$. Therefore, M is free, so $\operatorname{pr.dim}_A M = \operatorname{depth}_A M = 0$ and the formula is correct.

Suppose now that $d > 0$ and the formula holds for rings of depth $d - 1$. If $\operatorname{depth}_A M > 0$, there is a non-zero-divisor $a \in A$ which is also a non-zero-divisor on M. Then, by Corollaries 1.3 and 1.14

$$\begin{aligned}
\operatorname{depth}_{A/aA} A/aA &= d - 1, \\
\operatorname{depth}_{A/aA} M/aM &= \operatorname{depth}_A M - 1 \\
\operatorname{pr.dim}_{A/aA} M/aM &= \operatorname{pr.dim}_A M.
\end{aligned}$$

Since the formula holds for M/aM as A/aA-module by the inductive supposition, it holds for M too.

Finally, let $\operatorname{depth}_A M = 0$. There is an exact sequence $0 \to N \to P \to M \to 0$, where P is free. Then $\operatorname{pr.dim}_A M = \operatorname{pr.dim}_A N + 1$, while $\operatorname{depth}_A M = \operatorname{depth}_A N - 1$ by the Depth lemma. In particular, $\operatorname{depth}_A N = 1$, so the formula holds for N, whence it follows for M. □

1e **Theorem 1.15.** *Let M be a Cohen–Macaulay A-module of Krull dimension d, $\mathfrak{p} \in \operatorname{supp} M$ and m be the maximal length of M-sequences of elements from $\mathfrak{p}$. Then* $\dim A/\mathfrak{p} = d - m$ *and $M_{\mathfrak{p}}$ is a Cohen–Macaulay $A_{\mathfrak{p}}$-module of Krull dimension m. In particular, any localization of a Cohen–Macaulay ring is also Cohen–Macaulay.*

Proof. Choose an M-sequence $\mathbf{a} = (a_1, a_2, \dots, a_m)$ in $\mathfrak{p}$ and set $\overline{M} = M/\mathbf{a}M$; it is a Cohen–Macaulay modules of Krull dimension $d - m$. Then $\mathfrak{p} \in \operatorname{supp} \overline{M}$. Let $\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_k$ be all minimal prime ideals from $\operatorname{supp} \overline{M}$. Then $\operatorname{Ass}_A \overline{M} = \{\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_k\}$. If $\mathfrak{p} \neq \mathfrak{p}_i$ for all i, there is an element $a \in \mathfrak{p}$ which is a non-zero-divisor on $\overline{M}$. Then $(a_1, a_2, \dots, a_m, a)$ is an M-sequence in $\mathfrak{p}$, a contradiction. Therefore, $\mathfrak{p} = \mathfrak{p}_i$ for some i and $\operatorname{Kr.dim} A/\mathfrak{p} = d - m$ by Corolloary 1.9. On the other hand, since $\mathfrak{p}$ is a minimal prime ideal containing $\operatorname{ann}_A \overline{M}$, $\mathrm{Kr.dim}\, \overline{M}_{\mathfrak{p}} = 0$, so $\overline{M}_{\mathfrak{p}}$ is a Cohen–Macaulay $A_{\mathfrak{p}}$-module. Moreover $\mathbf{a}$ is an $M_{\mathfrak{p}}$-sequence in $A_{\mathfrak{p}}$. Therefore, $M_{\mathfrak{p}}$ is also a Cohen–Macaulay $A_{\mathfrak{p}}$-module of Krull dimension m. □

s2 2. Canonical modules and Gorenstein rings

Recall that the *injective dimension* of an A-module M is defined as

$$\mathrm{inj.dim}_A M = \sup \{ i \mid \operatorname{Ext}^i_A N, M \neq 0 \text{ for some } A\text{-module } N \}.$$

It is known that this value does not change if we only consider finitely generated modules N. Since every finitely generated A-module has a composition series with factors $A/\mathfrak{p}_i$, where $\mathfrak{p}_i \in \operatorname{spec} A$,

$$\mathrm{inj.dim}_A M = \sup \{ i \mid \operatorname{Ext}^i_A(A/\mathfrak{p}, M) \neq 0 \text{ for some } \mathfrak{p} \in \operatorname{spec} A \}.$$

One can even precise this formula.

21 **Lemma 2.1.** *Let $\mathfrak{p} \neq \mathfrak{m}$ be a prime ideal of A and M be an A-module such that $\operatorname{Ext}^{i+1}_A(A/\mathfrak{q}, M) = 0$ for any prime ideal $\mathfrak{q} \supset \mathfrak{p}$. Then $\operatorname{Ext}^i_A(A/\mathfrak{p}, M) = 0$.*

Proof. Choose an element $a \in \mathfrak{m} \setminus \mathfrak{p}$. It gives an exact sequence

$$0 \to A/\mathfrak{p} \xrightarrow{a} A/\mathfrak{p} \to A/(\mathfrak{p} + aA) \to 0,$$

whence an exact sequence

$$\operatorname{Ext}^i_A(A/\mathfrak{p}, M) \xrightarrow{a} \operatorname{Ext}^i_A(A/\mathfrak{p}, M) \to \operatorname{Ext}^{i+1}_A(A/(\mathfrak{p} + aA), M).$$

Since $A/(\mathfrak{p} + aA)$ has a composition series with factors $A/\mathfrak{q}$ for $\mathfrak{q} \supset \mathfrak{p}$, the third term of the last exact sequence is 0, so $a \operatorname{Ext}^i_A(A/\mathfrak{p}, M) = \operatorname{Ext}^i_A(A/\mathfrak{p}, M)$. By Nakayama's lemma, $\operatorname{Ext}^i_A(A/\mathfrak{p}, M) = 0$. □

22 **Corollary 2.2.** $\mathrm{inj.dim}_A M = \sup \{ i \mid \operatorname{Ext}^i_A(\mathbf{k}, M) \neq 0 \}$.

23 **Proposition 2.3.** *Let $\mathbf{a} = (a_1, a_2, \dots, a_m)$ be an A-sequence, $\overline{A} = A/\mathbf{a}$. Then $\mathrm{pr.dim}_A \overline{A} = m$ and $\operatorname{Ext}^m_A(\overline{A}, M) \simeq M/\mathbf{a}M$. In particular, if M is finitely generated, $\operatorname{Ext}^m_A(\overline{A}, M) \neq 0$.*

Proof. If $m = 1$, it follows from the exact sequence $0 \to A \xrightarrow{a} A \to \overline{A} \to 0$. Suppose that the claim holds for $A_1 = A/(a_1, a_2, \dots, a_{m-1})$. The exact sequence $0 \to A_1 \xrightarrow{a_m} A_1 \to \overline{A} \to 0$ gives the exact sequence

$$\operatorname{Ext}^{i-1}_A(A_1, M) \xrightarrow{a_m} \operatorname{Ext}^{i-1}_A(A_1, M) \to \operatorname{Ext}^i_M(\overline{A}, M) \to \operatorname{Ext}^i_A(A_1, M).$$

If $i > m$, it implies that $\operatorname{Ext}^i_A(\overline{A}, M) = 0$. If $i = m$, it gives

$$\operatorname{Ext}^m_A(\overline{A}, M) \simeq \operatorname{Ext}^{m-1}_A(A_1, M)/a_m \operatorname{Ext}^{m-1}_A(A_1, M) \simeq M/\mathbf{a}M.$$

□

24 **Theorem 2.4.** *If M is a finitely generated A-module and* $\mathrm{inj.dim}_A M < \infty$, *then*

$$\mathrm{Kr.dim}\, M \leqslant \mathrm{inj.dim}_A M = \mathrm{depth}_A A.$$

In particular, if A is a Cohen–Macaulay ring, then $\mathrm{inj.dim}_A M = \mathrm{Kr.dim}\, A$.

Proof. Let $k = \mathrm{inj.dim}_A M$, $d = \mathrm{depth}_A A$ and $\mathbf{a} = (a_1, a_2, \dots, a_d)$ be an A-sequence. Then $\mathrm{Ext}^d_A(A/\mathbf{a}, M) \neq 0$, so $k \geqslant d$. On the other hand, $\mathrm{depth}_A A/\mathbf{a} = 0$, so there is an embedding $\mathbf{k} \to A/\mathbf{a}$. It induces an epimorphism $\mathrm{Ext}^k_A(A/\mathbf{a}, M) \to \mathrm{Ext}^k_A(\mathbf{k}, M) \neq 0$. Therefore $d = \mathrm{pr.dim}_A A/\mathbf{a} \geqslant k$.

Let $m = \mathrm{Kr.dim}\, M$ and $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \dots \subset \mathfrak{p}_m = \mathfrak{m}$ be a maximal chain of prime ideals containing $\mathrm{ann}_A M$. Set $A_i = A_{\mathfrak{p}_i}$, $M_k = M_{\mathfrak{p}_i}$ and $\mathbf{k}_i = A_i/\mathfrak{p}_i A_i$, the residue field of A_i. We use induction on i to show that $\mathrm{Ext}^i_A(\mathbf{k}_i, M_i) \neq 0$. If $i = 0$, then $\mathfrak{p}_0 A_0$ is a unique prime in A_0 associated to M_0, so $\mathrm{Hom}_{A_0}(\mathbf{k}_0, M_0) \neq 0$. Let $i > 0$. Then

$$\mathrm{Ext}^{i-1}_{A_i}(A_i/\mathfrak{p}_{i-1}A_i, M_i)_{\mathfrak{p}_{i-1}} \simeq \mathrm{Ext}^{i-1}_{A_{i-1}}(\mathbf{k}_{i-1}, M_{i-1}) \neq 0$$

by the inductive supposition. Since $\mathfrak{p}_i A_i$ is a unique prime ideal of A_i containing $\mathfrak{p}_{i-1} A_i$, Lemma 2.1 implies that $\mathrm{Ext}^i_{A_i}(\mathbf{k}_i, M_i) \neq 0$. Now, for $i = m$, we get that $\mathrm{Ext}^m_A(\mathbf{k}, M) \neq 0$, whence $m \leqslant k$. □

25 **Definition 2.5.** (1) A maximal Cohen–Macaulay A-module ω is called *canonical*, if $\mathrm{inj.dim}_A \omega < \infty$ and $\mathrm{Ext}^d_A(\mathbf{k}, A) \simeq \mathbf{k}$, where $d = \mathrm{Kr.dim}\, A$.

(2) A is said to be *Gorenstein* if $\mathrm{inj.dim}_A A < \infty$.

Remark. Theorem 2.4 imlies that if a caninical module exists or if A is Gorenstein, then A is a Cohen–Macaulay ring. Moreover, any canonical module is maximal Cohen–Macaulay.

26 **Example 2.6.** (1) If A is regular, it is obviously Gorenstein. ~~Moreover, it is a canonical module itself.~~ Moreover, if $\mathfrak{m} = (a_1, a_2, \dots, a_d)$, where $d = \mathrm{Kr.dim}\, A$, then $(a_1, a_2, \dots, a_d)$ is an A-sequence, so $\mathrm{Ext}^d_A(\mathbf{k}, A) \simeq A/\mathfrak{m} = \mathbf{k}$, so A is a canonical module.

(2) If A is Artinian (or, the same, of finite length), it has a canonical module. It is an injective envelope $E(\mathbf{k})$ of the residue field $\mathbf{k}$. A is Gorenstein if and only if it is *Frobenius*, or *selfinjective*, i.e. A is an injective A-module.

(3) If A is a finite dimensional algebra over a field K (then $K \subseteq \mathbf{k}$), the functor $\mathrm{D} = \mathrm{Hom}_K(\_, K)$ is an exact duqality on the category A-mod. Therefore, the dual module $\mathrm{D}A$ is a canonical A-module.

The following lemma is a background for inductive proofs of results concerning injective dimension and canonical modules.

27 **Lemma 2.7.** *Let $a \in A$ be a non-zero-divisor both on A and on M, and $aN = 0$. Then*

$$\operatorname{Ext}^i_A(N, M) \simeq \operatorname{Ext}^{i-1}_{A/aA}(N, M/aM) \text{ for all } i > 0.$$

Proof. Modules N with $aN = 0$ are just modules over the ring $\bar{A} = A/aA$. From the exact sequence $0 \to M \xrightarrow{a} M \to M/aM \to 0$ we get the exact sequence

$$0 = \operatorname{Hom}_A(N, M) \to \operatorname{Hom}_A(N, M/aM) \to \operatorname{Ext}^1_A(N, M) \xrightarrow{a} \operatorname{Ext}^1_A(N, M).$$

Since the last map in this sequence is zero,

$$\operatorname{Ext}^1_A(N, M) \simeq \operatorname{Hom}_A(N, M/aM) = \operatorname{Hom}_{A/aA}(N, M/aM).$$

Thus $\operatorname{Ext}^i_A(\_, M)$, being the $(i-1)$-st left derived functor of $\operatorname{Ext}^1_A(\_, M)$ on the category of $\bar{A}$-modules, coincide with the $(i-1)$-st derived functor of $\operatorname{Hom}_{A/aA}(\_, M/aM)$, that is with $\operatorname{Ext}^{i-1}_{A/aA}(\_, M)$. □

28 **Corollary 2.8.** *Let an element $a \in A$ be a non-zero-divisor both on A and on M.*

(1) $\operatorname{inj.dim}_A M = \operatorname{inj.dim}_{A/aA} M/aM - 1$.

(2) *M is a canonical A-module if and only if M/aM is a canonical A/aA-module.*

(3) *Let M be maximal Cohen–Macaulay and $\mathbf{a} = (a_1, a_2, \dots, a_m)$ be an A-sequence. M is a canonical A-module if and only if $M/\mathbf{a}M$ is a canonical $A/\mathbf{a}$-module.*

29 **Theorem 2.9.** *If a ring A is Gorenstein, it is a canonical module over itself.*

Proof. Let $d = \operatorname{Kr.dim} A = \operatorname{depth}_A A$. We only have to prove that $\operatorname{Ext}^d_A(\mathbf{k}, A) \simeq \mathbf{k}$. Choose an A-sequence $\mathbf{a} = (a_1, a_2, \dots, a_d)$. By Lemma 2.7 and Corollary 2.8, the ring $\bar{A} = A/\mathbf{a}$ is Gorenstein of Krull dimension 0 and $\operatorname{Ext}^d_A(\mathbf{k}, A) \simeq \operatorname{Hom}_{\bar{A}}(\mathbf{k}, \bar{A})$. But $\bar{A}$ is an indecomposable injective $\bar{A}$-module, so it can only have one minimal submodule. Therefore, $\operatorname{Hom}_{\bar{A}}(\mathbf{k}, \bar{A}) \simeq \mathbf{k}$. □

20 **Theorem 2.10.** *Let ω be a canonical module for A, M be a Cohen–Macaulay A-module (not necessaryly maximal), $d = \operatorname{Kr.dim} A$ and $m = \operatorname{Kr.dim} M$.*

(1) $\operatorname{Ext}^i_A(M, \omega) = 0$ *for $i \neq d - m$.*

(2) $\operatorname{Ext}^{d-m}_A(M, \omega)$ *is a Cohen–Macaulay A-module of Krull dimension m.*

(3) *If M maximal Cohen–Macaulay, then* $\operatorname{Ext}^i_A(M,\omega) = 0$ *for* $i > 0$, $M^\vee = \operatorname{Hom}_A(M,\omega)$ *is also a maximal Cohen–Macaulay A-module and, for any A-sequence* $\mathbf{a} = (a_1, a_2, \dots, a_r)$,

e21 (2.1) $$M^\vee/\mathbf{a}M^\vee \simeq \operatorname{Hom}_{A/\mathbf{a}}(M/\mathbf{a}M, \omega/\mathbf{a}\omega).$$

Proof. We use induction on m. If $m = 0$, M is of finite length, hence $\operatorname{Ext}^i_A(M,\omega) = 0$ for $i \neq d$, and $\operatorname{Ext}^d_A(M,\omega)$ is also of finite length, that is Cohen–Macaulay of Krull dimension 0.

Let $m > 0$ and $a \in \mathfrak{m}$ bea non-zero-divisor on M. The exact sequence $0 \to M \xrightarrow{a} M \to M/aM \to 0$ induces the exact sequence

$$\operatorname{Ext}^i_A(M/aM,\omega) \to \operatorname{Ext}^i_A(M,\omega) \xrightarrow{a} \operatorname{Ext}^i_A(M,\omega) \to \operatorname{Ext}^{i+1}_A(M/aM,\omega).$$

Since M/aM is a Cohen–Macaulay A-module of Krull dimension $m-1$, the last term is zero if $i \neq d-m$ and is Cohen–Macaulay of Krull dimension $m-1$ if $i = d-m$. Therefore, $\operatorname{Ext}^i_A(M,\omega) = a\operatorname{Ext}^i_A(M,\omega)$ if $i \neq d-m$, whence $\operatorname{Ext}^i_A(M,\omega) = 0$. If $i = d-m$, we see that a is not a zero divisor on $\operatorname{Ext}^i_A(M,\omega)$ and $\operatorname{Ext}^i_A(M,\omega)/a\operatorname{Ext}^i_A(M,\omega) \simeq \operatorname{Ext}^{i+1}_A(M/aM,\omega)$, hence $\operatorname{Ext}^i_A(M,\omega)$ is Cohen–Macaulay of Krull dimension m.

In particular, if M is maximal Cohen–Macaulay, then so is $M^\vee$ and $\operatorname{Ext}^i_A(M,\omega) = 0$ for $i > 0$. If $a \in A$ is anon-zero-divisor, it is a non-zero-divisor on ω as well. The exact sequence $0 \to \omega \xrightarrow{a} \omega \to \omega/a\omega \to 0$ gives the exact sequence

$$0 \to \operatorname{Hom}_A(M,\omega) \xrightarrow{a} \operatorname{Hom}_A(M,\omega) \to \operatorname{Hom}_A(M,\omega/a\omega) \to 0,$$

whence

$$\begin{aligned} \operatorname{Hom}_A(M/aM,\omega/a\omega) &\simeq \operatorname{Hom}_A(M,\omega/a\omega) \simeq \\ &\simeq \operatorname{Hom}_A(M,\omega)/a\operatorname{Hom}_A(M,\omega). \end{aligned}$$

An immediate induction gives the formula (2.1). □

2a **Theorem 2.11.** *Let A be a Cohen–Macaulay ring.*

(1) *If ω and ω' are canonical modules for A, they are isomorphic.*
(2) *The natural map $A \to \operatorname{End}_A \omega$ is an isomorphism.*
(3) *The functor $M \mapsto M^\vee = \operatorname{Hom}_A(M,\omega)$ is an exact duality on the category* $\mathsf{CM}(A)$. *In particular, the natural map $\eta_M : M \to M^{\vee\vee}$ is an isomorphism.*

Proof. We use induction on $d = \operatorname{Kr.dim} A$. First suppose that $d = 0$, so $\mathsf{CM}(A) = A\text{-mod}$. Then ω is injective, so $^\vee$ is an exact functor. Moreover, $\mathbf{k}^\vee \simeq \mathbf{k}$, so the natural map $\mathbf{k} \to \mathbf{k}^{\vee\vee}$ is an isomorphim. An immediate induction on $\operatorname{length}_A M$ shows that $M \to M^{\vee\vee}$ is an isomorphism for all M. Therefore, $^\vee$ is an exact duality on A-mod.

In particular, since $\omega = A^\vee$, $\operatorname{End}_A \omega \simeq \operatorname{End}_A A \simeq A$ and $\operatorname{length}_A \omega = \operatorname{length}_A A = l$. Just in the same way, $\operatorname{length}_A \omega' = l$. Both ω and ω' have a unique minimal submodule isomorphic to $\mathbf{k}$. The embedding $\mathbf{k} \to \omega$ extends to a map $\phi : \omega' \to \omega$. Since $\operatorname{Ker} \phi$ does not contain the unique minimal submodule of ω', ϕ is a monomorphism, hence an isomorphism, since $\operatorname{length}_A \omega' = \operatorname{length}_A \omega$.

For the next considerations we use the following lemma.

2b **Lemma 2.12.** *Let $f : M \to N$ be a homomorphism of A-modules, N be a Cohen–Macaulay module and $\mathbf{a} = (a_1, a_2, \dots, a_m)$ be an A-sequence. f is an isomorphism if and only if so is the induced map $\bar{f} : M/\mathbf{a}M \to N/\mathbf{a}N$.*

Proof. Obviously, we only have to prove the claim for $m = 1$, so $\mathbf{a} = (a)$. If $\bar{f}$ is an isomorphism, f is an epimorphism by Nakayama's lemma, so we have the exact sequence $0 \to K \to M \xrightarrow{f} N \to 0$, where $K = \operatorname{Ker} f$. Since N is Cohen–Macaulay, a is not a zero divisor on N, hence $\operatorname{Tor}_1^A(A/aA, N) = 0$. Therefore, the sequence

$$0 \to K/aK \to M/aM \xrightarrow{\bar{f}} N/aN \to 0$$

is also exact. Since $\bar{f}$ is an isomorphism, $K/aK = 0$, so $K = 0$ by Nakayama's lemma and f is an isomorphism. □

Return to the proof of Theorem 2.11. Let now $d > 0$. Choose an element $a \in \mathfrak{m}$ which is a non-zero-divisor. Then both $\omega/a\omega$ and $\omega'/a\omega'$ are canonical A/aA-modules and $\operatorname{Kr.dim} A/aA = d-1$. By the inductive supposition, $\omega/a\omega \simeq \omega'/a\omega'$. Choose an isomorphism

$$\bar{\phi} \in \operatorname{Hom}_A(\omega'/a\omega', \omega/a\omega) \simeq \operatorname{Hom}_A(\omega, \omega')/a \operatorname{Hom}_A(\omega, \omega')$$

and its preimage $\phi \in \operatorname{Hom}_A(\omega, \omega')$. By Lemma 2.12, ϕ is an isomorphism. Just the same observation shows that the map $A \to \operatorname{Hom}_A(\omega, \omega)$ is an isomorphism. In particular, it means that η_A is an isomorphism. Let $M \in \mathsf{CM}(A)$. There is an exact sequence $P_1 \to P_0 \to M \to 0$, where P_i are free modules. Since kernels of epimorphisms of maximal Cohen–Macaulay modules are Cohen–Macaulay by the Depth lemma, and $\operatorname{Ext}_A^1(N, \omega) = 0$ for any maximal Cohen–Macaulay A-module N, the second dual sequence $P_1^{\vee\vee} \to P_0^{\vee\vee} \to M^{\vee\vee} \to 0$ is also exact, and since the natural maps $\eta_{P_i} : P_i \to P_i^{\vee\vee}$ are isomorphisms, so is the map $\eta_M : M \to M^{\vee\vee}$. □

Now we may (and will) speak about *the* canonical A-module and denote it by ω_A.

2c **Proposition 2.13.** *If ω is a canonical A-module, then $\omega_{\mathfrak{p}}$ is a canonical $A_{\mathfrak{p}}$-module for any $\mathfrak{p} \in \operatorname{spec} A$. In particular, if A is Gorenstein, so is $A_{\mathfrak{p}}$.*

Proof. We use induction on $d = \mathrm{Kr.dim}\, A$, the case $d = 0$ being obvious. If $d > 0$, choose a non-zero-divisor $a \in A$. Then $\overline{\omega} = \omega/a\omega$ is a canonical module over $\overline{A} = A/aA$ and $\mathrm{Kr.dim}\, \overline{A} = d-1$. By the inductive supposition, $\overline{\omega}_{\mathfrak{p}} = \omega_{\mathfrak{p}}/a\omega_{\mathfrak{p}}$ is a canonical module over $\overline{A}_{\mathfrak{p}} = A_{\mathfrak{p}}/aA_{\mathfrak{p}}$. By Corollary 2.8, $\omega_{\mathfrak{p}}$ is a canonical module over $A_{\mathfrak{p}}$. □

s3 3. Finite algebras

In this section we suppose that B is a *finite A-algebra*, i.e. an A-algebra which is finitely generated as A-module. We denote by $\iota = \iota_B : A \to B$ the natural homomorphism of rings mapping $a \in A$ to $a1 \in B$. We also suppose that B is local with the maximal ideal $\mathfrak{n}$ and the residue field $\mathfrak{b} = B/\mathfrak{n}$. Note that $\mathfrak{n} \supseteq \mathfrak{m}B$ and $\mathfrak{b}$ is a finite extension of the field $\mathbf{k}$.

31 **Proposition 3.1.** $\operatorname{depth}_A M = \operatorname{depth}_B M$ *for any finitely generated B-module M. In particular, M is a Cohen–Macaulay A-module if and only if it is a Cohen–Macaulay B-module.*

Proof. We use induction by $d = \operatorname{depth}_A M$, the case $d = 0$ being obvious. If $d > 0$, choose $a \in A$ which is a non-zero-divisor on M. Then $\operatorname{depth}_A M/aM = d-1$, so $\operatorname{depth}_B M/aM = d-1$ and $\operatorname{depth}_B M = d$. □

32 **Theorem 3.2.** *Let A and B be Cohen–Macaulay rings, $d = \mathrm{Kr.dim}\, A$, $c = \mathrm{Kr.dim}\, B$. If ω_A is a canonical A-module, then $\omega_B = \operatorname{Ext}_A^{d-c}(\omega_A, B)$ is a canonical B-module. Moreover, $\operatorname{Ext}_B^p(N, \omega_B) \simeq \operatorname{Ext}_A^{p+d-c}(N, \omega_A)$ for any B-module N; in particular, if N is a maximal Cohen–Macaulay B-module, then $N^\vee \simeq \operatorname{Ext}_A^{d-c}(N, \omega_A)$.*

Proof. By Theorem 2.10, ω_B is a maximal Cohen–Macaulay B-module and $\operatorname{Ext}_A^i(B, \omega_A) = 0$ for $i \neq d-c$. Since

$$\operatorname{Hom}_A(N, M) \simeq \operatorname{Hom}_B(N, \operatorname{Hom}_A(B, M))$$

for every A-module M and B-module N, there is a spectral sequence

$$\operatorname{Ext}_B^p(N, \operatorname{Ext}_A^q(B, M)) \Rightarrow \operatorname{Ext}_A^{p+q}(N, M).$$

If $M = \omega_A$, $\operatorname{Ext}_A^q(B, \omega_A) = 0$ for $q \neq d-c$, so this spectral sequence degenerates to isomorphisms $\operatorname{Ext}_B^p(N, \omega_B) \simeq \operatorname{Ext}_A^{p+d-c}(N, \omega_A)$. Therefore, $\mathrm{inj.dim}_B\, \omega_B < \infty$. If $N = \mathbf{b}$, we get $\operatorname{Ext}_B^c(\mathbf{b}, \omega_B) \simeq \operatorname{Ext}_A^d(\mathbf{b}, \omega_A) \simeq \mathbf{k}^r$, where $r = \dim_{\mathbf{k}} \mathbf{b}$. Since $\operatorname{Ext}_B^c(\mathbf{b}, \omega_B)$ is a $\mathbf{b}$-module, it implies that $\operatorname{Ext}_B^c(\mathbf{b}, \omega_B) \simeq \mathbf{b}$, so ω_B is indeed a canonical B-module. □

33 **Corollary 3.3.** (1) *If a Cohen–Macaulay ring A is a finite algebra over a Gorenstein ring (in particular, over a regular ring), it has a canonical module.*

(2) *If a Cohen–Macaulay ring A is a localization of a finitely generated algebra over a field, it has a canonical module.*

(3) *If a Cohen–Macaulay ring A is* complete, *it has a canonical module.*

(4) *A Cohen–Macaulay analytic algebra has a canonical module.*

We recall that an *analytic algebra* over a topological field $\mathbf{k}$ is a quotient of the algebra $\mathbf{k}\{x_1, x_2, \dots, x_n\}$ of the *convergent power series*, i.e. power series that are convergent in a neighbourhood of zero.

Proof. (1) is a partial case of Theorem 3.2.

(2) follows from (1) and the Noether normalization lemma [13, Chapter III.D, Theorem 2].

(3) follows from (1) and the fact that every complete local Noetherian ring is a finite algebra over a regular local ring [2, Chapitre IX].

(4) follows from (1) and the fact that $\mathbf{k}\{x_1, x_2, \dots, x_n\}$ is regular. □

s4 4. Surface singularities

In this section we suppose that A is a reduced ring of Krull dimension 2 (a *surface singularity*). We denote by Q the total ring of fractions of A, by $\varepsilon_\mathfrak{p}$ the natural homomorphism $Q \to Q_\mathfrak{p}$ and for every torsion free A-module M set

$$M^\dagger = \{\, a \in Q \otimes_A M \mid (\varepsilon_\mathfrak{p} \otimes 1)(a) \in M_\mathfrak{p} \text{ for all } \mathfrak{p} \in \mathrm{sp}_1 A \,\},$$

where $\mathrm{sp}_1 A = \{\, \mathfrak{p} \in \operatorname{spec} A \mid \operatorname{ht} \mathfrak{p} = 1 \,\}$ and we identify $M_\mathfrak{p}$ with its image in $Q_\mathfrak{p} \otimes M$. Then $M \subseteq M^\dagger$ (we identify M with $1 \otimes M \subset Q \otimes_A M$). If A is *integral*, i.e. without zero divisors, then $\varepsilon_\mathfrak{p}$ is an isomorphism identifying Q with $Q_\mathfrak{p}$ and, under these identifications, $M^\dagger = \bigcap_{\mathfrak{p} \in \mathrm{sp}_1 A} M_\mathfrak{p}$. If M is not torsion free, we denote by $\operatorname{tors} M$ its torsion part and set $M^\dagger = (M/\operatorname{tors} M)^\dagger$.

First we prove the following properties.

41 **Proposition 4.1.** *If M is a maximal Cohen–Macaulay A-module, so is also* $\operatorname{Hom}_A(N, M)$ *for any A-module N.*

Proof. Consider an exact sequence $A^m \xrightarrow{f} A^n \to N \to 0$. Applying $\operatorname{Hom}_A(\_, M)$, we get an exact sequence

$$0 \to \operatorname{Hom}_A(N, M) \to M^n \xrightarrow{f^*} M^m \to \operatorname{Coker} f^* \to 0.$$

Now the claim follows from Corollary 1.11. □

42 **Theorem 4.2.** (1) *If M is a torsion free A-module, the following conditions are equivalent:*

(a) *M is maximal Cohen–Macaulay.*

(b) $M = \cancel{\bigcap_{\mathfrak{p}\in \mathrm{sp}_1 A} M_{\mathfrak{p}}}$ $M^{\dagger}$

(2) *If A is normal, i.e. integral and integrally closed in Q, it is Cohen–Macaulay, and the conditions above are also equivalent to the next one:*

(c) *M is* reflexive, *i.e. the natural map $M \to M^{**}$ is an isomorphism, where $M^* = \mathrm{Hom}_A(M, A)$.*

(Actually, the last claim is also valid if A is *Gorenstein in codimension* 1, i.e. all localizations $A_{\mathfrak{p}}$ for $\mathfrak{p} \in \mathrm{sp}_1 A$ are Gorenstein.)

(3) *If A has a canonical module ω_A, these conditions are also equivalent to the next one:*

(d) *The natural map $\eta_M : M \to M^{\vee\vee}$ is an isomorphism, where $M^{\vee} = \mathrm{Hom}_A(M, \omega_A)$.*

Proof. (1) Suppose that $M \neq M^{\dagger}$. Since $(M^{\dagger}/M)_{\mathfrak{p}} = 0$ for each $\mathfrak{p} \in \mathrm{sp}_1 A$, $\mathrm{supp}_A M^{\dagger}/M = \{\mathfrak{m}\}$ and there is a submodule in $M^{\dagger}/M$ isomorphic to $\mathbf{k}$. Its preimage N in $M^{\dagger}$ gives a nonsplit exact sequence $0 \to M \to N \to \mathbf{k} \to 0$, so $\mathrm{Ext}^1_A(\mathbf{k}, M) \neq 0$ and M is not Cohen–Macaulay.

On the contrary, let $\mathrm{Ext}^1_A(\mathbf{k}, M) \neq 0$, so there is a nonsplit exact sequence $0 \to M \to N \to \mathbf{k} \to 0$. Then N is also torsion free, $Q \otimes_A M = Q \otimes_A N$ and $M_{\mathfrak{p}} = N_{\mathfrak{p}}$ for all $\mathfrak{p} \in \mathrm{sp}_1 A$. Therefore, $M^{\dagger} = N^{\dagger} \supseteq N$ and $M^{\dagger} \neq M$. Hence, the conditions (a) and (b) are indeed equivalent.

(2) If A is normal, then $A = A^{\dagger}$ and all $A_{\mathfrak{p}}$, where $\mathfrak{p} \in \mathrm{sp}_1 A$, are discrete valuation rings [13, Section III.C, § 1]. Then torsion free $A_{\mathfrak{p}}$-modules are free, so the natural maps $M_{\mathfrak{p}} \to M_{\mathfrak{p}}^{**}$ are isomorphims for every torsion free A-module M. Since M^{**} is Cohen–Macaulay by Proposition 4.1, it implies that (c)$\Leftrightarrow$(b).

(3) is proved similarly. □

43 **Corollary 4.3.** *A homomorphism $f : M \to N$ of maximal Cohen–Macaulay A-modules is an isomorphism if and only if so are its localizations $f_{\mathfrak{p}} : M_{\mathfrak{p}} \to N_{\mathfrak{p}}$ for all $\mathfrak{p} \in \mathrm{sp}_1 A$.*

44 **Corollary 4.4.** (1) *The embedding of categories $\mathsf{CM}(A) \to A$-mod has a left adjoint functor $M \mapsto M^{\dagger}$.*

(2) *If A is normal (or at least Gorenstein in codimension 1), then $M^{\dagger} \simeq M^{**}$.*

(3) *If A has a canonical module, then $M^{\dagger} \simeq M^{\vee\vee}$.*

Proof. It follows from Theorem 4.2, Proposition 4.1 and the fact that $\mathrm{Hom}_A(N, M)_{\mathfrak{p}} = \mathrm{Hom}_{A_{\mathfrak{p}}}(N_{\mathfrak{p}}, M_{\mathfrak{p}}) = \mathrm{Hom}_{A_{\mathfrak{p}}}(N^{\dagger}_{\mathfrak{p}}, M^{\dagger}_{\mathfrak{p}}) = \mathrm{Hom}_A(N^{\dagger}, M^{\dagger})_{\mathfrak{p}}$ for all $\mathfrak{p} \in \mathrm{sp}_1 A$. □

The module $M^{\dagger}$ is often called the *Macaulayfication* of M.

45 **Corollary 4.5.** *Let M be an A-module. The functor* $\mathrm{Hom}_A(M, \_) : \mathsf{CM}(A) \to A\text{-mod}$ *has a left adjoint* $\_ \boxtimes_A M = (\_ \otimes_A M)^{\dagger}$.

We apply the last result to the extensions of Cohen–Macaulay rings.

46 **Definition 4.6.** Let $A \subseteq B$ be an extension of rings. A homomorphism of A-modules $\rho : B \to A$ is said to be a *Reynolds erator* if $\rho(a) = a$ for all $a \in A$.

Obviously, a Reynolds operator exists if and only if $B = A \oplus C$ for some A-submodule $C \subset B$.

46 **Example 4.7.** Suppose that a finite group G of order n acts on a ring B and g is invertible in B. Let A be the subring of *invariants*, i.e.

$$A = B^G = \{\, a \in B \mid \sigma(a) = a \text{ for all } \sigma \in G \,\}.$$

Then

$$\rho(r) = \frac{1}{n} \sum_{\sigma \in G} \sigma(r)$$

is a Reynolds operator.

47 **Proposition 4.8.** *Let $A \subseteq B$ be an extension of rings such that there is a Reynolds operator $\rho : B \to A$.*

(1) *If I is an ideal of A, then $IB \cap I = I$.*
(2) *If B is Noetherian, so is A.*
(3) *If B is local, so is A.*
(4) *If B is Cohen–Macaulay and integral over A, then A is Cohen–Macaulay as well.*

Proof. (1) If $a = \sum_i a_i b_i$, where $a \in A$, $a_i \in I$, $b_i \in B$, then $\rho(a) = \sum_i a_i \rho(b_i) \in I$.

(2) If $I_0 \subset I_1 \subset I_2 \subset \dots$ is a chain of ideals in A, then $I_0 B \subset I_1 B \subset I_2 B \subset \dots$ is a chain of ideals in B, so it must be finite.

(3) Let $\mathfrak{n}$ be the unique maximal ideal in B, $\mathfrak{m} = A \cap \mathfrak{n}$. For every ideal $I \subset A$, $IB \subseteq \mathfrak{m}$, whence $I = IB \cap A \subset \mathfrak{m}$.

(4) We keep the previous notations and use the induction on $d = \operatorname{depth}_A B$. If $d = 0$, then all elements of $\mathfrak{m}$ are zero divisors in B. Since B is Cohen–Macaulay, it means that $\mathfrak{m} \subseteq \bigcup_{i=1}^{m} \mathfrak{p}_i$, where $\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_m$ are minimal prime ideals of B. Therefore, $\mathfrak{m} \subseteq \mathfrak{p}_i$ for some i, whence

$\mathfrak{m} = A \cap \mathfrak{p}_i$. Since B is integral over A, $\mathfrak{p}_i$ is a maximal ideal in B and $\mathrm{Kr.dim}\, A = \mathrm{Kr.dim}\, B = 0$, so A is Cohen–Macaulay.

If $d > 0$ and $a \in A$ is a non-zero-divisor in B, then $aB \cap A = \rho(aB) = aA$, so $A/aA \subseteq B/aB$, $\mathrm{depth}_{A/aA} B/aB = d-1$ and ρ induces a Reynolds operator $B/aB \to A/aA$. Since B/aB is Cohen–Macaulay, so is A/aA by the inductive suposition, hence also A/aA. □

kahn 5. Kahn reduction

From now on let $\mathbf{k}$ be an algebraically closed field, A be a complete noetherian normal local $\mathbf{k}$-algebra of Krull dimension 2 with the maximal ideal $\mathfrak{m}$ such that $A/\mathfrak{m} = \mathbf{k}$. We call such an algebra a *normal surface singularity*. We set $S = \operatorname{spec} A$, $\check{S} = S \setminus \{\mathfrak{m}\}$.

k1 **Definitions and Notations 5.1.** For a normal surface singularity we denote:

(1) $\pi : X \to S$ a *resolution* of A, i.e. a projective morphism such that X is smooth and the restriction of π onto $\check{X} = \pi^{-1}\check{S}$ is an isomorphism $\check{X} \to \check{S}$.
(2) $E = \pi^{-1}(\mathfrak{m})_{\mathrm{red}}$, the *exceptional curve* of the resolution π. It is a connected projective curve.
(3) $E_1, E_2, \dots, E_s$ the connected components of E and $C(X)$ the subgroup in the group $\mathrm{Div}(X)$ of divisors on X generated by $E_1, E_2, \dots, E_s$. If $Z = \sum_i k_i E_i$ with all $k_i \geqslant 0$, we call Z an *exceptional cycle* on X. We always consider such a cycle as a (nonreduced) projective, identifying it with the subscheme of X defined by the sheaf of ideals $\mathcal{O}_X(-Z)$.
(4) $(\_.\_)$ the intersection form on the group $\mathrm{Div}\, X$ of divisors on X. It is known [11, 12] that the its restriction onto $C(X)$ is negative definite.
(5) $b_i = -(E_i.E_i)$ and $g_i = p_a(E_i) = \dim_{\mathbf{k}} \mathrm{H}^1(E_i, \mathcal{O}_{E_i})$, the geometrical genus of E_i.
(6) ω_X the canonical sheaf on X and $K = K_X$ its class in $\mathrm{Div}\, X$. Recall that, by the adjunction formula, $(K.E_i) = b_i + 2g_i - 2$ [8].
(7) For a coherent sheaf $\mathcal{F}$ on X we set $\mathcal{F}^\vee = \mathcal{H}om_X(\mathcal{F}, \mathcal{O}_X)$. It is a locally free sheaf on X.
(8) We identify coherent sheaves on S with finitely generated A-modules and define, for such a module M, $\pi^\dagger M = (\pi^* M)^{\vee\vee}$.
(9) We denote by $\mathsf{VB}(V)$ the category of locally free (coherent) sheaves (or, the same, vector bundles) on a scheme V.
(10) For a coherent sheaf $\mathcal{F}$ on a scheme V we denote by $\mathcal{F}^0$ its subsheaf generated by the global sections, i.e. the image of the

evaluation map

$$\mathrm{ev}_{\mathcal{F}} : \mathrm{Hom}_X(\mathcal{O}_X, \mathcal{F}) \otimes \mathcal{O}_X \to \mathcal{F}.$$

If $\mathcal{F}^0 = \mathcal{F}$, we say that $\mathcal{F}$ is *globally generated*. We say that $\mathcal{F}$ is *generically globally generated* if $\operatorname{supp} \mathcal{F}/\mathcal{F}^0$ is 0-dimensional.

k2 **Theorem–Definition 5.2.** *Let* VB^0 *be the full subcategory of* $\mathsf{VB}(X)$ *consisting of the sheaves isomorphic to* $\pi^\dagger M$, *where* $M \in \mathsf{CM}(A)$. *We call locally free sheaves from* VB^0 full locally free sheaves.

(1) *The functors* $\pi^\dagger : \mathsf{CM}(A) \to \mathsf{VB}^0(X)$ *and* $\pi_* : \mathsf{VB}^0(X) \to \mathsf{CM}(A)$ *are quasi-inverse to each, thus establishing an equivalence of the categories* $\mathsf{CM}(A)$ *and* $\mathsf{VB}^0(X)$.

(2) *A locally free sheaf* $\mathcal{F} \in \mathsf{VB}(X)$ *is full if and only if*

(a) $\mathcal{F}$ *is generically globally generated.*

full2 (b) *The restriction map* $\mathrm{H}^0(X, \mathcal{F}) \to \mathrm{H}^0(\check{X}, \mathcal{F})$ *is surjective (hence bijective).*

The latter condition can be reformulated as follows:

full3 (c) *The natural map* $\gamma_{\mathcal{F}} : \mathrm{H}^1_E(X, \mathcal{F}) \to \mathrm{H}^1(X, \mathcal{F})$ *is injective.*

Proof. We start from(2). Let $M \in \mathsf{CM}(A)$, $\mathcal{F}' = \pi^* M/\operatorname{tors}(\pi^* M)$ and $\mathcal{F} = \pi^\dagger M = (\mathcal{F}')^{\vee\vee}$. Then $\mathcal{F}'$ is globally generated and $\mathcal{F}'_x$ is free for all x outside 0-dimensional subset $Y \subset X$. Therefore, $\mathcal{F}'_x = \mathcal{F}_x$ for $x \notin Y$, so $\mathcal{F}$ is generically globally generated. Moreover, $Y \cap \check{X} = \emptyset$, so $\mathrm{H}^0(\check{X}, \mathcal{F}) = \mathrm{H}^0(\check{X}, \mathcal{F}') = \mathrm{H}^0(\check{S}, M)$. On the other hand, $\mathrm{H}^0(X, \pi^* M) = \mathrm{H}^0(S, M) = M$ and the map $\mathrm{H}^0(\check{S}, M) \to M$ is an isomorphism since M is maximal Cohen–Macaulay. It proves (b).

Suppose that (a) and (b) hold and set $M = \pi_* \mathcal{F}$. Then the map $\mathrm{H}^0(S, M) = \mathrm{H}^0(X, \mathcal{F}) \to \mathrm{H}^0(\check{S}, M) = \mathrm{H}^0(\check{X}, \mathcal{F})$ is bijective, hence $M \in \mathsf{CM}(A)$. Moreover, $\mathcal{F}' = \pi^* M/\operatorname{tors}(\pi^* M) \subseteq \mathcal{F}$ is globally generated, and $\mathrm{H}^0(X, \mathcal{F}') = M = \mathrm{H}^0(X, \mathcal{F})$. Hence, $\mathcal{F}' = \operatorname{Im} \mathrm{ev}_{\mathcal{F}}$ Since $\mathcal{F}$ is generically globally generated, $\operatorname{supp}(\mathcal{F}/\mathcal{F}')$ is 0-dimensional and $\mathcal{F} \simeq (\mathcal{F}')^{\vee\vee} = \pi^\dagger 9M$.

Let now $\mathcal{F} = \pi^\dagger M$, $\mathcal{G} = \pi^\dagger N$. We have already seen that $M = \pi_* \mathcal{F}$ and $N = \pi_* \mathcal{G}$. Since $\mathcal{G}$ is locall free, $\mathrm{Hom}_X(\pi^* M, \mathcal{G}) \simeq \mathrm{Hom}_X(\pi^\dagger M, \mathcal{G})$. Hence $\mathrm{Hom}_A(M, \pi_* \mathcal{G}) \simeq \mathrm{Hom}_X(\pi^* M, \mathcal{G}) \simeq \mathrm{Hom}_X(\pi^\dagger M, \mathcal{G})$, so $\pi^\dagger$ is right adjoint to π_*. As the natural maps $M \to \pi_* \mathcal{F}$ and $\pi^\dagger M \to \mathcal{F}$ are isomorphisms, $\pi^\dagger$ and π_* are quasi-inverse to each other.

The equivalence of (b) and (c) follows from the exact sequence for local cohomologies

$$0 \to \mathrm{H}^0_E(X, \mathcal{F}) \to \mathrm{H}^0(X, \mathcal{F}) \to \mathrm{H}^0(\check{X}, \mathcal{F}) \to \mathrm{H}^1_E(X, \mathcal{F}) \to \mathrm{H}^1(X, \mathcal{F})$$

□

If Z is an exceptional cycle on X, $i_Z : Z \to X$ its embedding, we denote by $R_Z : \mathsf{CM}(A) \to \mathsf{VB}(Z)$ the composition $i_Z^*\pi^\dagger$ and call R_Z the *Kahn reduction functor* with respect to Z.

k3 **Definition 5.3.** Let Z be an exceptional cycle on X.

(1) We call a locally free sheaf F on Z *full* if $\mathsf{F} \simeq \mathcal{F}_Z = i_Z^*\mathcal{F}$ for some full locally free sheaf $\mathcal{F}$ on X. We denote by $\mathsf{VB}^0(Z)$ the full subcategory of $\mathsf{VB}(Z)$ consisting of full locally free sheaf.

(2) We call Z a *weak reduction cycle* if the following conditions hold:
  (a) $\mathcal{O}_Z(-Z)$ is generically globally generated.
  (b) $\mathrm{H}^1(E, \mathcal{O}_Z(-Z)) = 0$.
  If, moreover,
  (c) $\omega_Z^\vee = \mathcal{H}om_Z(\omega_Z, \mathcal{O}_Z)$ is generically globally generated,
  where $\omega_Z = i_Z^*\omega_X(Z)$ is the canonical sheaf on Z, we call Z a *reduction cycle.*

k4 **Proposition 5.4.** *A reduction cycle always exists.*

Proof. Since the intersection from is negative definite, there is an exceptional cycle C such that $(C.E_i) < 0$ for all i. Then $\mathcal{O}_C(-C)$ is ample on C [11, Proposition 10.4]. Therefore, the cycle nC satisfies the conditions (a–c) for n big enough. □

k-main **Theorem 5.5.** (1) *Let Z be a weak reduction cycle. A locally free sheaf F on Z is full if and only if*
  (a) *F is generically globally generated.*
  (b) *There is an extension F_2 of F to a locally free sheaf on $2Z$ such that the boundary map $\delta_{\mathsf{F}} : \mathrm{H}^0(E, \mathsf{F}(Z)) \to \mathrm{H}^1(E, \mathsf{F})$ induced by the Z-twist $0 \to \mathsf{F} \to \mathsf{F}_2(Z) \to \mathsf{F}(Z) \to 0$ of the natural exact sequence $0 \to \mathsf{F}(-Z) \to \mathsf{F}_2 \to \mathsf{F} \to 0$ is injective.*

(2) *If Z is a reduction cycle, F a full locally free sheaf on Z. There is a unique full locally free sheaf $\mathcal{F}$ on X such that $\mathsf{F} = \mathcal{F}_Z$.*

Therefore, if Z is a reduction cycle, the Kahn reduction functor $R_Z = i_Z^*\pi^\dagger : \mathsf{CM}(A) \to \mathsf{VB}^0(Z)$ induces a one-to-one correspondence between isomorphim classes of maximal Cohen–Macaulay A-modules and full locally free sheaf on Z.

The proof of this theorem consists of several steps. In what follows we fix a weak reduction cycle Z and denote by i_n the embedding $nZ \to X$, $\mathcal{O} = \mathcal{O}_Z$ and $\mathcal{O}_n = \mathcal{O}_{nZ}$. For a locally free sheaf $\mathcal{F}$ on X we set $\mathsf{F}_n = i_n^*\mathcal{F}$; in particular, $\mathsf{F} = \mathsf{F}_1$. It is a locally free sheaf on nZ and there are exact sequences

Sn (S$_n$) $$0 \to \mathsf{F}_n(-Z) \to \mathsf{F}_{n+1} \to \mathsf{F} \to 0.$$

Its Z-twist for $n = 1$ induces the boundary map $\delta_{\mathsf{F}} : \mathrm{H}^0(E, \mathsf{F}(Z)) \to \mathrm{H}^1(E, \mathsf{F})$ of the condition (b) of Theorem 5.5. For any n, twisting S_n by $(n+1)Z$, we obtain an exact sequence

Sn1 (S$'_n$) $$0 \to \mathsf{F}_n(nZ) \to \mathsf{F}_{n+1}((n+1)Z) \to \mathsf{F}((n+1)Z) \to 0.$$

It induces homomorphisms

$$\lambda_n^i : \mathrm{H}^i(E, \mathsf{F}_n(nZ)) \to \mathrm{H}^i(E, \mathsf{F}_{n+1}((n+1)Z)),$$

so we can consider their dirtect limit.

k6 **Lemma 5.6** (Wahl's lemma, [14, Lemma B.2]). *For any $i \geqslant 0$*

$$\varinjlim_n \mathrm{H}^i(E, \mathsf{F}_n(nZ)) \simeq \mathrm{H}_E^{i+1}(X, \mathcal{F}).$$

Proof. The exact sequence $0 \to \mathcal{O}_X(-nZ) \to \mathcal{O}_X \to \mathcal{O}_n \to 0$ implies that $\mathcal{E}xt_X^i(\mathcal{O}_n, \mathcal{F}) = 0$ for $i \neq 1$ and $\mathcal{E}xt_X^1(\mathcal{O}_n, \mathcal{F}) \simeq \mathcal{F}(nZ)/\mathcal{F} \simeq \mathsf{F}_n(nZ)$. Therefore, the spectral sequence for Ext gives isomorphisms $\mathrm{Ext}_X^{i+1}(\mathcal{O}_n, \mathcal{F}) \simeq \mathrm{H}^i(X, \mathsf{F}_n(nZ))$. Obviously, the diagram

$$\begin{array}{ccc} \mathrm{Ext}_X^{i+1}(\mathcal{O}_n, \mathcal{F}) & \xrightarrow{\sim} & \mathrm{H}^i(X, \mathsf{F}_n(nZ)) \\ {\scriptstyle \phi_n^*}\big\downarrow & & \big\downarrow{\scriptstyle \lambda_n^i} \\ \mathrm{Ext}_X^{i+1}(\mathcal{O}_{n+1}, \mathcal{F}) & \xrightarrow{\sim} & \mathrm{H}^i(X, \mathsf{F}_{n+1}(n+1)) \end{array},$$

induced by the natural map $\phi_n : \mathcal{O}_{n+1} \to \mathcal{O}_n$, is commutative. Since, by [5, Theorem 2.8],

$$\mathrm{H}_E^{i+1}(X, \mathcal{F}) \simeq \varinjlim_n \mathrm{Ext}^{i+1}(\mathcal{O}_n, \mathcal{F}),$$

it gives us the necessary isomorphism. □

We denote by $\widehat{\mathrm{H}}^i(X, \mathcal{F})$ the $\mathfrak{m}$-adic completion of the A-module $\mathrm{H}^i(X, \mathcal{F})$. Note that $\operatorname{supp} \mathrm{H}^1(X, \mathcal{F}) = \{\mathfrak{m}\}$ and all $\mathrm{H}^i(X, \mathcal{F})$ are finitely generated A-modules, since π is projective. Therefore $\widehat{\mathrm{H}}^i(X, \mathcal{F})$ is of finite length, so $\widehat{\mathrm{H}}^1(X, \mathcal{F}) = \mathrm{H}^1(X, \mathcal{F})$. Recall also that $\mathrm{H}^i(X, \mathcal{F}) = 0$ for $i > 1$ [8, Corollary III.11.2], so H^1 is left exact.

k7 **Lemma 5.7.** *If F is generically globally generated, then so is also $\mathcal{F}$ and $\mathrm{H}^1(X, \mathcal{F}(-Z)) = 0$.*

In particular, the conditions (a) of Theorem 5.2 and of Theorem 5.5 are equivalent.

Proof. By the theorem on formal functions [8, Theorem III.11.1],

$$\mathrm{H}^1(X,\mathcal{F}(-Z)) = \widehat{\mathrm{H}}^1(X,\mathcal{F}(-Z)) \simeq \varprojlim_n \mathrm{H}^1(E,\mathsf{F}_n(-Z)).$$

Exact sequences (S_n) show that F_n has a chain of subsheaves with factors $\mathsf{F}(-mZ)$ for $0 \leqslant m < n$, hence $\mathsf{F}_n(-Z)$ has a chain of subsheaves with factors $\mathsf{F}(-(m+1)Z)$. Since F and $\mathcal{O}_Z(-Z)$ are generically globally generated, so are all sheaves $\mathsf{F}(-mZ)$, so there are exact sequences

$$r\mathcal{O} \to \mathsf{F}(-mZ) \to \mathcal{T} \to 0$$

and

$$r\mathcal{O}(-Z) \to \mathsf{F}(-(m+1)Z) \to \mathcal{T} \to 0,$$

where $\operatorname{supp}\mathcal{T}$ is 0-dimensional, so $\mathcal{T}(-Z) \simeq \mathcal{T}$. Since $\mathrm{H}^1(E,\mathcal{O}(-Z)) = 0$ and $\mathrm{H}^1(E,\mathcal{T}) = 0$, it gives that $\mathrm{H}^1(E,\mathsf{F}(-(m+1)Z)) = 0$. Therefore, $\mathrm{H}^1(E,\mathsf{F}_n(-Z)) = 0$ and $\mathrm{H}^1(X,\mathcal{F}(-Z)) = 0$.

Note that the resriction on $\check{X}$ of the sheaf $\mathcal{F}$ coincide with that of $\pi^*\pi_*\mathcal{F}$, which is globally generated. Therefore $\operatorname{supp}\operatorname{Coker}\mathrm{ev}_{\mathcal{F}} \subseteq E$. The exact sequence

$$0 \to \mathcal{F}(-Z) \to \mathcal{F} \to \mathsf{F} \to 0$$

shows that the map $\mathrm{H}^0(X,\mathcal{F}) \to \mathrm{H}^0(X,\mathsf{F})$ is surjective, i.e. every section of F lifts to a section of $\mathcal{F}$. Since $\mathcal{F}(-Z)_x \subseteq \mathfrak{m}_x\mathcal{F}_x$ for any $x \in E$ and F is generically globally generated, the sheaf $\mathcal{F}$ is also generically globally generated. □

k8 **Corollary 5.8.** *An exceptional cycle Z is a weak reduction cycle if and only if $\mathcal{O}_X(-Z)$ is generically globally generated and $\mathrm{H}^1(X,\mathcal{O}_X(-Z)) = 0$. It is a reduction cycle if and only if also $\omega_X^\vee(-Z)$ is generically globally generated.*

Proof. The necessity follows from Lemma 5.7. Suppose that $\mathcal{O}_X(-Z)$ is generically globally generated and $\mathrm{H}^1(X,\mathcal{O}_X(-Z)) = 0$. There is a homomorphism $f : m\mathcal{O}_X \to \mathcal{O}_X(-Z)$ with 0-dimensional $\operatorname{supp}\operatorname{Coker} f$. Then $\operatorname{supp}\operatorname{Coker} f(-Z)$ is also 0-dimensional, so there is a surjection $\mathrm{H}^1(X,m\mathcal{O}_X(-Z)) \to \mathrm{H}^1(X,\mathcal{O}_X(-2Z))$, whence $\mathrm{H}^1(X,\mathcal{O}_X(-2Z)) = 0$. Therefore, the exact sequence

$$0 \to \mathcal{O}_X(-2Z) \to \mathcal{O}_X(-Z) \to \mathcal{O}_Z(-Z) \to 0$$

induces a surjection $\mathrm{H}^0(X,\mathcal{O}_X(-Z)) \to \mathrm{H}^0(X,\mathcal{O}_Z(-Z))$ and an isomorphim $\mathrm{H}^1(X,\mathcal{O}_X(-Z)) \simeq \mathrm{H}^1(X,\mathcal{O}_Z(-Z))$. Thus Z is a weak reduction cycle. Now the last claim also follows from Lemma 5.7, since $\omega_Z^\vee \simeq \omega_X^\vee(-Z) \otimes_{\mathcal{O}_X} \mathcal{O}_Z$. □

k9 **Proposition 5.9.** *Let $\mathcal{F}$ be a locally free sheaf on X. It is full if and only if* F *is generically globally generated and the map* δ_{F} *is injective.*

Proof. By Lemma 5.7, we only have to prove that $\gamma_{\mathcal{F}}$ is injective if and only if so is δ_{F}. By Wahl's lemma, we have a commutative diagram

ek1 (5.1)
$$\begin{array}{ccc} \mathrm{H}^0(E,\mathsf{F}(Z)) & \xrightarrow{\delta_{\mathsf{F}}} & \mathrm{H}^1(E,\mathsf{F}) \\ \downarrow & & \uparrow \\ \mathrm{H}^1_E(X,\mathcal{F}) & \xrightarrow{\gamma_{\mathcal{F}}} & \mathrm{H}^1(X,\mathcal{F}), \end{array}$$

where the left vertical map is injective, while the right vertical map is bijective, since $\mathrm{H}^1(X,\mathcal{F}(-Z))=0$ by Lemma 5.7. Therefore, if $\gamma_{\mathcal{F}}$ is injective, so is δ_{F}.

Let now δ_{F} is injective. We will show that in this case the maps

$$\lambda_n=\lambda_n^0:\mathrm{H}^0(E,\mathsf{F}_n(nZ))\to\mathrm{H}^0(E,\mathsf{F}_{n+1}((n+1)Z))$$

arizing from the exact sequences (S'_n) are surjective, hence bijective. Then the left vertical map int he diagram 5.1 is bijective too, so $\gamma_{\mathcal{F}}$ is injective.

Note that λ_n is surjective if and only if the boundary map $\beta_n:\mathrm{H}^0(E,\mathsf{F}((n+1)Z))\to\mathrm{H}^1(E,\mathsf{F}_n(nZ))$ is injective. Denote by α_n the composition of β_n with the restriction $\mathrm{H}^1(E,\mathsf{F}_n(nZ))\to\mathrm{H}^1(E,\mathsf{F}(nZ))$. It is the boundary map for the exact sequence

Sn2 (S″) $$0\to\mathsf{F}(nZ)\to\mathsf{F}_2((n+1)Z)\to\mathsf{F}((n+1)Z)\to 0,$$

especially, $\alpha_0=\delta_{\mathsf{F}}$ is injective. Since $\mathcal{O}_X(-Z)$ is generically globally generated, so is also $\mathcal{O}_X(-nZ)$, hence there is an exact sequence $m\mathcal{O}_X\to\mathcal{O}_X(-nZ)\to\mathcal{M}\to 0$ with 0-dimensional $\operatorname{supp}\mathcal{M}$. It gives an exact sequence $m\mathcal{O}_n\to\mathcal{O}_n(-nZ)\to\mathcal{M}_{nZ}\to 0$. Dualizing, we obtain a monomorphism $\mathcal{O}_X(nZ)\to m\mathcal{O}_X$ such that the induced maps $\mathcal{O}_n(nZ)\to m\mathcal{O}_n$ are monomorphisms too. Tensoring it with (S″), we obtain a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathsf{F}(nZ) & \longrightarrow & \mathsf{F}_2((n+1)Z) & \longrightarrow & \mathsf{F}((n+1)Z) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & m\mathsf{F} & \longrightarrow & m\mathsf{F}_2(Z) & \longrightarrow & \mathsf{F}(Z) & \longrightarrow & 0, \end{array}$$

where all vertical maps are injective. It induces a commutative diagram

$$\begin{array}{ccc} \mathrm{H}^0(\mathsf{F}((n+1)Z) & \xrightarrow{\alpha_n} & \mathrm{H}^1(\mathsf{F}(nZ)) \\ \downarrow & & \downarrow \\ m\mathrm{H}^0(\mathsf{F}(Z)) & \xrightarrow{m\alpha_0} & m\mathrm{H}^1(sF), \end{array}$$

where the left vertical map is injective. Since α_0 is injective too, so is α_n for all n, hence also β_n. It accomplishes the proof. □

Recall some results on noncommutative cohomologies and their relation to locally free sheaves [6, 7]. The locally free sheaves of a prescribed rank r on the scheme nZ are in one-to-one correspondence with the elements of the cohomology set $\mathrm{H}^1(G_n)$, where $G_n = \mathrm{GL}(r, \mathcal{O}_n)$. The exact sequence

$$0 \to \mathcal{O}(-nZ) \to \mathcal{O}_{n+1} \to \mathcal{O}_n \to 0$$

gives rise to the exact sequence of group bundles

$$0 \to \mathcal{N} \to G_{n+1} \to G_n \to 0,$$

where $\mathcal{N} = \mathcal{H}om_{nZ}(\mathcal{O}_n, \mathcal{O}(-nZ)) \simeq \mathcal{O}_n^\vee \otimes \mathcal{O}$. Therefore, there is an exact sequence of punctured sets

ek2 (5.2) $$\mathrm{H}^0(G_n) \xrightarrow{\delta} \mathrm{H}^1(\mathcal{N}) \xrightarrow{f} \mathrm{H}^1(G_{n+1}) \xrightarrow{g} \mathrm{H}^1(G_n).$$

Here "exact" means that at each place the preimage of the neutral element coincides with the image of the preceeding map. Moreover, for any element $c \in \mathrm{H}^1(G_n)$ one can naturally define an element $\partial c \in \mathrm{H}^2(\mathcal{N}^c)$, where $\mathcal{N}^c$ is the twist of $\mathcal{N}$ with respect to c, so that $c \in \operatorname{Im} g$ if and only if $\partial c = 0$. The fibres of f coincide with the orbits of the natural action of $\mathrm{H}^0(G_n)$ on $\mathrm{H}^1(\mathcal{N})$. Namely, if $g \in \mathrm{H}^0(G_n)$, choose an affine covering $\{U_i\}$ of E such that $g|_{U_i} = g_i$ for some $g_i \in \Gamma(U_i, G_{n+1})$. Then $\delta(g) = (g_i^{-1}g_j$. If $\gamma \in \mathrm{H}^1(\mathcal{N})$n is presented by the cocycle (γ_{ij}) on this covering, then $g \cdot \gamma = \delta(g) + (g_i\gamma_{ij}g_j^{-1})$.

Proof of Theorem 5.5(1). Since H^2 vanishes on E, the map g in (5.2) is surjective, i.e. every locally free sheaf $\mathcal{F}_n$ on nZ lifts to a locally free sheaf $\mathcal{F}_{n+1}$ on $(n+1)Z$. Thus any locally free sheaf F on Z lifts to a locally free sheaf F_n on nZ, and these liftings can be so chosen that, for every n, the sheaf F_n coincides with the resriction of F_{n+1}. Then the inverse limit of F_n gives a locally free sheaf $\mathcal{F}$ on X. Moreover, if F is generically globally generated and δ_{F} is injective, then $\mathcal{F}$ is full by Proposition 5.9, which proves the claim. □

Since $\mathrm{H}^1(\mathcal{O}(-Z)) = 0$ and $\mathrm{H}^1(\mathcal{O}(-Z))$ is generically globally generated, then also $\mathrm{H}^1(\mathcal{O}(-nZ)) = 0$ (see the proof of Lemma (5.7)). Therefore, every section of G_n lifts to a section of G_{n+1} and $\operatorname{Im} \delta = 0$. Hence, the action of $\mathrm{H}^0(G_n) \subset \mathrm{Hom}_{nZ}(\mathcal{O}_n, \mathcal{O}_n)$ on $\mathrm{H}^1(\mathcal{N})$ coincide with the Yoneda multiplication.

k0 **Lemma 5.10.** *Let $\mathcal{F}$ be a locally free sheaf on X. Then $\mathrm{H}^1_Z(\mathcal{F}) \simeq \mathrm{H}^1(\mathcal{F}^\vee \otimes \omega_X)'$, where V' denotes the dual vector space to V. Moreover,* *these isomorphisms are compatible with those of Lemma 5.6 and the Grothendieck duality, i.e. the arising diagrams*

ek3 (5.3)
$$\begin{tikzcd} \mathrm{H}^0(\mathsf{F}_n(nZ)) \arrow[r, "\sim"] \arrow[d] & \mathrm{H}^1(\mathsf{F}_n^\vee(-nZ) \otimes \omega_{nZ})' \arrow[d] \\ \mathrm{H}^1_Z(\mathcal{F}) \arrow[r, "\sim"] & \mathrm{H}^1(\mathcal{F}^\vee \otimes \omega_X)' \end{tikzcd}$$

are commutative.

Here the right vertical map comes by duality from the epimorphism $\mathcal{F} \to \mathsf{F}_n$ and the isomorphim $\omega_{nZ} \simeq \mathcal{O}_n \otimes \omega_X(nZ)$. Note that both vertical mas are isomorphisms: the left one by Lemma 5.6 and the right one since $\mathrm{H}^2 = 0$ on X, so H^1 is right exact.

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