$\boxed{K1}$

Lecture 3

Drozd

Definitions + Notations

$\pi : X \to S \qquad \check{X} \xrightarrow{\sim} \check{S}$

M MCM $\iff$ the restriction $\Gamma(\check{\check{S}}, M) \to \Gamma(\check{S}, M)$ is iso

$E = E_1 \cup \dots \cup E_s \qquad E_1, \dots, E_s$ are prime divisors on X

$M \longmapsto M^{\#} = (\pi^* M)^{\vee\vee} \qquad \mathcal{F}^{\vee} = \mathcal{H}om_X(\mathcal{F}, \mathcal{O}_X)$

locally free

$\boxed{\text{in this case } M \simeq \pi_* \mathcal{F}}$

Theorem $\quad \mathcal{F} \simeq \pi^{\#} M \iff$

generic epi ~~ggg~~

(1) $\mathcal{F}$ is ggg !

(2) $H^0(X, \mathcal{F}) \longrightarrow H^0(\check{X}, \mathcal{F})$ is surjective

$(\Rightarrow)$ $\mathcal{F}' = \pi^* M / \text{torsion} \subset \mathcal{F} \qquad (\mathcal{F} = (\mathcal{F}')^{\vee\vee})$

torsion free $\Rightarrow$ $\mathcal{F}'_x$ free ~~at x~~ for x not closed !

$\Rightarrow$ ~~free outside fin many points~~ Supp $(\mathcal{F}/\mathcal{F}')$ 0-dim.

$$\left\{ \begin{array}{ccc} H^0(\check{X}, \mathcal{F}) & = & H^0(\check{S}, M) \\ \text{surj} \uparrow & & \wr\wr \\ H^0(X, \mathcal{F}) & \longleftarrow & H^0(S, M) \end{array} \right\}$$

$(\Leftarrow)$ Set $M = \pi_* \mathcal{F}$

torsion free

$$\begin{array}{ccc} H^0(S, M) & = & H^0(X, \mathcal{F}) \\ \downarrow & & \downarrow \text{surj.} \\ H^0(\check{S}, M) & = & H^0(\check{X}, \mathcal{F}) \end{array} \Rightarrow \text{iso}$$

$\mathcal{F}' = \pi^* M / \text{torsion} \subset \mathcal{F}$ and has the same global sections (since any "global section" of M $=$ of $\mathcal{F}$ gives a global section of $\mathcal{F}'$)

$\Rightarrow$ $\mathcal{F}/\mathcal{F}'$ has 0-dim support $\Rightarrow (\mathcal{F}')^{\vee\vee} = \mathcal{F}$

$\square$

Reinterpretation of (2) — local cohomology

$$H^0_E(X,\mathcal{F}) = \{ s \in H^0(X,\mathcal{F}) \mid s|_{\check{X}} = 0\}$$

$$= \varinjlim \mathrm{Hom}_X(\mathcal{O}_X/\mathcal{I}^m, \mathcal{F})$$

$\boxed{\mathcal{I} = \mathcal{I}(E)}$ $H^i_E(X,\mathcal{F})$ = i-th right deriv. of H^0

$$= \varinjlim \mathrm{Ext}^i_X(\mathcal{O}_X/\mathcal{I}^m, \mathcal{F})$$

Main exact sequence

$$0 \to H^0_E(X,\mathcal{F}) \to H^0(X,\mathcal{F}) \to H^0(\check{X},\mathcal{F}) \to$$
$$\to H^1_E(X,\mathcal{F}) \to H^1(X,\mathcal{F}) \to H^1(\check{X},\mathcal{F}) \to \cdots$$

So (2) means that $H^1_E(X,\mathcal{F}) \to H^1(X,\mathcal{F})$ is inj

Intersection theory on ~~a closed curve C on~~ X

Omit!

$\mathcal{F}$ lfs on X, C a positive divisor (= closed curve)

$\Longrightarrow \quad \mathcal{F}_C = \mathcal{F} \otimes \mathcal{O}_C$

$\chi(\mathcal{F}_C) = \dim H^0(\mathcal{F}_C) - \dim H^1(\mathcal{F}_C)$

$\deg \mathcal{F}_C = \chi(\mathcal{F}_C) - r\chi(\mathcal{O}_C)$ [motivated by R-R for $r=1$!]

D another divisor $\rightsquigarrow \mathcal{O}_X(D) \rightsquigarrow \mathcal{O}_C(D)$

[$(D.C) := \deg_C \mathcal{O}_C(D)$]

it is symmetric bilinear (easy)

[Lipman Publ. Math. IHES 36 (1969)] $\Longrightarrow$

+ (Mumford) restricted on divisors with support E it is $\underbrace{\text{negative definite}}_{\mathrm{Div}_E X}$

"exceptional div."

+ "adjunction formula"

[K3] (L3

Adj. f.: $\quad \boxed{\mathrm{F}} \quad \chi(\mathcal{O}_Z) = -\frac{1}{2}(K+Z.Z)$

in part. $\quad (K+E_i.E_i) = 2g_i - 2 \qquad g_i =$ arithmet. genus of E_i

later!

$(K.E_i) = b_i + 2g_i - 2 \qquad b_i = -(E_i.E_i) > 0$

in part. $\quad (K.E_i) > 0$ except the case

$g_i = 0,\ b_i = 2 \quad = 0$

$(E_i \simeq \mathbb{P}^1) \quad b_i = 1 \quad < 0 \quad \Leftarrow$ "contractable line"

(no such lines on the minimal resolution!)

Kahn's reduction functor

$\Rrightarrow \quad R_Z M = \mathrm{res}_Z M^\#$

Questions: 1) Image ?

2) When does R_Z reflect isomorphisms ?

Reduction cycle $\quad Z$ positive exceptional div. is said to be a reduction cycle if

(1) $\mathcal{O}_Z(-Z)$ is ggg

(2) $H^1(\mathcal{O}_Z(-Z)) = 0$

$\Big\}$ "weak red.c."

(3) $\omega_Z^\vee = \mathcal{H}om_Z(\omega_Z, \mathcal{O}_Z)$ is ggg

[$\omega_Z = \omega_X \otimes \mathcal{O}_Z(Z)$ is the canonical sheaf on Z giving the Serre-Groth. duality

$$H^i(\mathcal{F}) \simeq \mathrm{Ext}_Z^{1-i}(\mathcal{F}, \omega_Z)' \qquad \text{(dual space)}$$

Since the intersection form is negative on $\mathrm{Div}_E X$, such Z always exists: take C s.t. $(C.E_i) < 0$, then Serre's Theorems imply that nC is red. cycle for n big enough

Suppose F l.f.s. on Z

F_2 — " — on $2Z$ s.t. $(F_2)_Z = F$

~~Then~~ $0 \to F(-Z) \to F_2 \to F \to 0$

$\rightsquigarrow 0 \to F \to F_2(Z) \to F(Z) \to 0$

$\rightsquigarrow \partial_F : H^0(F(Z)) \to H^1(F)$

Kahn Thm (Math. Ann. 285 (1989) 141–160)

① Z wrc $\Rightarrow$ equiv:

(i) $F \in \operatorname{Im} R_Z$

(ii) F ggg and $\exists F_2$ on $2Z$ s.t. ∂_F is injective $\not\Rightarrow$ A

② Z r.c., $M \not\cong M' \Rightarrow R_Z M \not\cong R_Z M'$

A: In what follows, Z w.r.c.

$\mathcal{F}$ loc. free on X, $\mathcal{F}_n = \mathcal{F}_{nZ}$

Lemma $\mathcal{F}_Z$ ggg $\Rightarrow$ $\mathcal{F}$ ggg $\wedge$ $H^1(\mathcal{F}(-Z)) = 0$

Pf $H^1(\mathcal{F}(-Z)) \cong \varprojlim H^1(\mathcal{F}_n(-Z))$ "thm on formal functions"

$\mathcal{F}_n$ has a chain of subsheaves with the factors $\mathcal{F}(-mZ)$, $0 \le m < n$

$\boxed{\mathcal{F}_n(-Z) \quad 0 < m \le n}$

$\mathcal{O}_Z(-Z)$, $\mathcal{F}_Z$ ggg $\Rightarrow \mathcal{F}_Z(-mZ)$ ggg $\forall m \ge 0$

$\Rightarrow \exists$ gen. epi $r\mathcal{O}_Z \to \mathcal{F}_Z(-(m-1)Z)$ $(m>0)$

$\Rightarrow$ ~~epi~~ $r\mathcal{O}_Z(-Z) \to \mathcal{F}_Z(-mZ)$

$\Rightarrow$ epi $\underbrace{H^1(r\mathcal{O}_Z(-Z))}_{0} \to H^1(\mathcal{F}_Z(-mZ))$ $(m>0)$

$\nearrow 0$

$H^1(\mathcal{F}(-Z)) = 0$

(A) Z : $\mathcal{O}_Z(-Z)$ ggg, $H^1(\mathcal{O}_Z(-Z))=0$ core (L3

1. $\mathcal{F}_Z$ ggg $\Rightarrow$ $\mathcal{F}$ ggg and $H^1(\mathcal{F}(-Z))=0$

Pf 1) $H^1(\mathcal{F}(-Z)) = \varprojlim H^1(\mathcal{F}_{nZ}(-Z))$

$$\left[\ 0 \to \mathcal{F}_{nZ}(-Z) \to \mathcal{F}_{(n+1)Z} \to \mathcal{F}_Z \to 0\ \right] \quad (!)$$

$\Rightarrow$ $\mathcal{F}_{nZ}(-Z)$ has a filtration with the factors $\mathcal{F}_Z(-mZ)$ $1 \le m \le n$

$\mathcal{F}_Z, \mathcal{O}_Z(-Z)$ ggg $\Rightarrow$ $\mathcal{F}_Z(-(m-1)Z)$ ggg $(m \geq 0)$

$\Rightarrow$ $\exists$ $r\mathcal{O}_Z \to \mathcal{F}_Z(-(m-1)Z)$ gen. surj

$r\mathcal{O}_Z(-Z) \to \mathcal{F}_Z(-mZ)$ —"—

$\Rightarrow$ $\underbrace{H^1(r\mathcal{O}_Z(-Z))}_{0} \twoheadrightarrow H^1(\mathcal{F}_Z(-mZ))$

$\Rightarrow$ $\boxed{H^1(\mathcal{F}_{nZ}(-Z))=0}$ $\Rightarrow$ $H^1(\mathcal{F}(-Z))=0$

2) $0 \to \mathcal{F}(-Z) \to \mathcal{F} \to \mathcal{F}_Z \to 0$ +

~~$\cancel{H^0}$~~ $H^0(\mathcal{F}) \twoheadrightarrow H^0(\mathcal{F}_Z)$

$\mathcal{F}(-Z)_x \subseteq \mathfrak{m}_x \mathcal{F}_x$ for $x \in E$

$\Rightarrow$ $\boxed{\mathcal{F} \text{ ggg}}$

So Z core $\Rightarrow$ $\mathcal{O}_X(-Z)$ ggg and $H^1(\mathcal{O}_X(-Z))=0$

Moreover, $\mathcal{F}$ ggg $\Leftrightarrow$ $\mathcal{F}_Z$ ggg

(B) Consider the condition:

$$H^1_E(\mathcal{F}) \longrightarrow H^1(\mathcal{F}) \text{ mono}$$

(L3

Lemma (Wahl) $\quad H^1_E(\mathcal{F}) \simeq \varinjlim H^0(\mathcal{F}_{nZ}(nZ))$

by natural map (inj!) coming from (!) twisted by $(n+1)Z$

Pf $\quad H^1_E(\mathcal{F}) = \varinjlim \mathrm{Ext}^1_X(\mathcal{O}_{nZ}, \mathcal{F})$

$$0 \to \mathcal{O}_X(-nZ) \to \mathcal{O}_X \to \mathcal{O}_{nZ} \to 0 \qquad \mathcal{H}om_X(-, \mathcal{F})$$

$$0 = \mathcal{H}om_X(\mathcal{O}_{nZ}, \mathcal{F}) \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}(nZ) \to \mathcal{E}xt^1_X(\mathcal{O}_{nZ}, \mathcal{F}) \to 0$$

$$\Rightarrow \quad \mathcal{E}xt^1_X(\mathcal{O}_{nZ}, \mathcal{F}) \simeq \boxed{\mathcal{F}(nZ)/\mathcal{F} \simeq \mathcal{F}_{nZ}(nZ)}$$

$$\Rightarrow \quad \mathrm{Ext}^1_X(\mathcal{O}_{nZ}, \mathcal{F}) \simeq H^0(\mathcal{F}_{nZ}(nZ)) \qquad \boxtimes$$

Theorem $\quad \mathcal{F}$ full $\Leftrightarrow$ $\mathcal{F}_Z$ sgg

and $\delta : H^0(\mathcal{F}_Z(Z)) \to H^1(\mathcal{F}_Z)$ inj.

$\boxed{\text{coming from } \mathcal{F}_{2Z}}$

Proof

$\mathcal{F}$ full $\Rightarrow$ sgg $\Rightarrow$ $\mathcal{F}_Z$ sgg

$$\begin{array}{ccc} H^0(\mathcal{F}_Z(Z)) & \xrightarrow{\delta} & H^1(\mathcal{F}_Z) \\ \downarrow & & \uparrow \\ H^1_E(\mathcal{F}) & \hookrightarrow & H^1(\mathcal{F}) \end{array}$$

$\leftarrow$ inj. since $H^1(\mathcal{F}(-Z)) = 0$

$\Rightarrow$ δ inj

Back

(c) L3

$$0 \to H^0(\mathcal{F}_{nZ}(nZ)) \to H^0(\mathcal{F}_{(n+1)Z}((n+1)Z)) \to H^0(\mathcal{F}_Z(nZ)) \xrightarrow{\partial_n}$$
$$\to H^1(\mathcal{F}_{nZ}(nZ))$$

<u>Claim:</u> <u>δ_n mono</u> $\Rightarrow$ ~~all~~ $H^1_E(\mathcal{F}) = H^0(\mathcal{F}_Z(Z))$

and

$$\begin{array}{ccc} H^0(\mathcal{F}_Z(Z)) & \xrightarrow{\delta} & H^1(\mathcal{F}_Z) \\ \| & & \uparrow \\ H^1_E(\mathcal{F}) & \xrightarrow[\beta]{} & H^1(\mathcal{F}) \end{array}$$

δ inj $\Rightarrow$ β inj.

Compare ∂_n with $H^1(\mathcal{F}_{nZ}(nZ)) \to H^1(\mathcal{F}_Z(nZ))$

get δ_n : $H^0(\mathcal{F}_Z((n+1)Z)) \xrightarrow{\delta} H^1(\mathcal{F}_Z(nZ))$

$\boxed{\text{which is } nZ\text{-twist of } \delta}$

Note that $\mathcal{O}_X(-nZ)$ ggg.

get $r\mathcal{O}_X \longrightarrow \mathcal{O}_X(-nZ)$ gen. epi.

giving semi-epi $r\mathcal{O}_{nZ} \to \mathcal{O}_Z(-nZ)$

$\mathcal{H}om(-, \mathcal{F}_Z)$ gives <u>mono</u>

$\mathcal{F}_Z(nZ) \hookrightarrow r\mathcal{F}_Z$

so

$$\begin{array}{ccccccc} 0 \to & \mathcal{F}_Z(nZ) & \to & \mathcal{F}_{2Z}((n+1)Z) & \to & \mathcal{F}_Z((n+1)Z) & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & r\mathcal{F}_Z & \to & r\mathcal{F}_{2Z}(Z) & \to & r\mathcal{F}_Z(Z) & \to 0 \end{array}$$

$$\rightsquigarrow \quad \begin{array}{ccc} H^0(\mathcal{F}_Z((n+1)Z)) & \xrightarrow{\delta_n} & H^1(\mathcal{F}_Z(nZ)) \\ \downarrow & & \downarrow \\ H^0(r\mathcal{F}_Z(Z)) & \xhookrightarrow{r\delta} & H^1(r\mathcal{F}_Z) \end{array}$$

$\Rightarrow \delta_n$ mono!