Cohen-Macaulay Modules, Surface Singularities and McKay correspondence

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Surface singularities I

1. Holomorphic functions and sheaves

$U \subset \mathbb{C}^n$ open, $f: U \to \mathbb{C}$ is holomorphic $\iff$

Notation:

$\forall\, p \in U \quad \exists\ V = V(p)$ (i.e. V is open neighb. of p)

$\exists\ g \in \mathbb{C}\{x_1 - p_1, \dots, x_n - p_n\}$ convergent power series in V

s.t. $f|_V = g$

$\mathcal{O}(U) := \mathcal{O}_{\mathbb{C}^n}(U) := \{ f: U \to \mathbb{C} \text{ holom} \}$ is a ring, even a $\mathbb{C}$-algebra

$\mathcal{O}_{\mathbb{C}^n}$ = sheaf of holomorphic functions on $\mathbb{C}^n$

Note: A sheaf $\mathcal{F}$ on a top. space X is a collection

$\{ \mathcal{F}(U), \rho^U_V \}_{U \subset X \text{ open},\ V \subset U}$ s.t.

i) $\mathcal{F}$ is an abelian group (a ring ...)

(ii) $\rho^U_V: \mathcal{F}(U) \to \mathcal{F}(V)$ is a group hom., $\rho^V_W \circ \rho^U_V = \rho^U_W$ & V

(iii) $\forall U$ open covering $\{U_i\}$ of U and $s_i \in \mathcal{F}(U_i)$ s.t.

$s_i|_{U_i \cap V_j} = s_j|_{U_i \cap V_j} \quad \exists!\ s \in \mathcal{F}(U),\ s|_{U_i} = s_i$

$s|_V := \rho^U_V(s)$

Note if $\mathcal{F}$ satisfies (i) & (ii) it is called a presheaf to wich we can associate a sheaf $\hat{\mathcal{F}}$, $\mathcal{F} = \hat{\mathcal{F}}$ if $\mathcal{F}$ is a sheaf

$\hat{\mathcal{F}}(U) = \{ s: U \to \coprod_{x \in U} \mathcal{F}_x \mid s(x) \in \mathcal{F}_x\ \&\ \exists V(x) \subset U,\ t \in \mathcal{F}(V),\ t_y = s(y)\ \forall y \in V \}$

$\Gamma(U, \mathcal{F}) := \hat{\mathcal{F}}(U)$.

For $x \in X$ set the stalk of $\mathcal{F}$ at x

$$\mathcal{F}_x := \coprod_{U = U(x)} \mathcal{F}(U) \,/\!\sim \qquad s \in \mathcal{F}(U) \sim t \in \mathcal{F}(V) \iff \exists\, W \subset U \cap V \text{ s.t. } s|_W = t|_W$$

$s_x :=$ class of $s \in \mathcal{F}(U)$ is the germ of s at x

$s \in \mathcal{F}(U)$ is a section of $\mathcal{F}$ over U

We have for each presheaf $\mathcal{F}_x = \hat{\mathcal{F}}_x$

- $\mathcal{O}_{\mathbb{C}^n}$ is a sheaf with ρ^U_V the restriction map

$\mathcal{C}^\infty_{\mathbb{R}^n}$ is the sheaf of C^∞ differentiable functions

etc.

2. Ringed spaces

Let $f: X \to Y$ be continuous map between top. spaces

- Let $\mathcal{F}$ a sheaf on X. Define the direct image sheaf

$f_* \mathcal{F}$ on Y by

$$(f_* \mathcal{F})(V) := \mathcal{F}(f^{-1}(V)) \qquad V \subset Y \text{ open}$$

- Let $\mathcal{G}$ be a sheaf on Y. Define the topological preimage sheaf

$f^{-1}\mathcal{G}$ on X by

$$(f^{-1}\mathcal{G})(\mathcal{U}) := \coprod_{V = V(f(U))} \mathcal{G}(V) / \sim \;= \varinjlim_{V = V(f(U))} \mathcal{G}(V)$$

with $\sim$ as above.

<u>Note</u>: $(f^{-1}\mathcal{G})_x = \mathcal{G}_{f(x)}$, $(f_* \mathcal{F})_y = \varinjlim_{V = V(y)} \mathcal{F}(f^{-1}V) \to \mathcal{F}_x \quad \forall x \in f^{-1}(y)$

<u>Def</u>: A ringed space is a pair $(X, \mathcal{A}_X)$ with X a top. space, $\mathcal{A}_X$ a sheaf of rings on X; $\mathcal{A}_X$ is $\mathbb{C}$-analytic ringed space $\iff \mathcal{A}_{X,x} \cong \mathbb{C}\{x_1 \dots x_n\}/I$ as local $\mathbb{C}$-algebra) i.e. $\mathcal{A}_{X,x}$ is an analytic local ring i.e.

<u>Example</u>: $(\mathbb{C}^n, \mathcal{O}_{\mathbb{C}^n})$ is a ringed space, even a $\mathbb{C}$-analytic ringed space;

A morphism of ringed spaces is a pair $(f, f^\#)$ with

$f: X \to Y$ continuous

$f^\#: \mathcal{A}_Y \to f_* \mathcal{A}_X$ a morph of sheaves of rings

<u>Notation</u> $\quad X$ instead of $(X, \mathcal{A}_X)$

$\qquad f$ instead of $(f, f^\#)$

(equiv. $\hat{f}: f^{-1}\mathcal{A}_Y \to \mathcal{A}_X$, $f^\#_x = \hat{f}_x : \mathcal{A}_{Y,f(x)} \to \mathcal{A}_{X,x} \quad \forall x \in X$

A sheaf $\mathcal{F}$ of $\mathcal{A}_X$-modules on X is called coherent $\iff$

- $\mathcal{F}$ is finite i.e. $\forall x \in X \;\exists\, \mathcal{U}(x)$ & a surjection $\mathcal{A}_X^p|_{\mathcal{U}} \twoheadrightarrow \mathcal{F}|_{\mathcal{U}}$ i.e. $\mathcal{A}_y^p \to \mathcal{F}_y$ is surj. $\forall y \in \mathcal{U}$
- $\mathcal{F}$ is relation finite: i.e. $\forall\, \mathcal{U} \subset X$ open and $\forall$ morph $\varphi: \mathcal{A}_X^p|_{\mathcal{U}} \to \mathcal{F}|_{\mathcal{U}}$, $\mathrm{Ker}(\varphi)$ is finite $((\mathrm{Ker}\,\varphi)_y = \mathrm{Ker}\,\varphi_y)$

Example:

$$\begin{tikzcd} \mathbb{C}^n \arrow[d, phantom, "\cup"] & \mathbb{C}^m \arrow[d, phantom, "\cup"] \\ \mathcal{U} \arrow[r, "f"] \arrow[dr, "g \circ f"'] & V \arrow[d, "g"] \\ & \mathbb{C} \end{tikzcd}$$

hol. $f = (f_1, \dots, f_m)$
$f_i \in \mathcal{O}_{\mathbb{C}^n}(\mathcal{U})$

$f^\# : \mathcal{O}_V \to f_* \mathcal{O}_{\mathcal{U}}$

$\underset{\mathbb{C}^m}{g \in \mathcal{O}(V)} \longrightarrow \mathcal{O}_{\mathbb{C}^n}(f^{-1}V)$

$f^\#(g) : f^{-1}(V) \longrightarrow \mathbb{C}$ $\qquad \forall x \in f^{-1}(V)$

$$f^\#_x : \mathcal{O}_{\mathbb{C}^m, f(x)} \to \mathcal{O}_{\mathbb{C}^n, x} \qquad f^\#_x(g_x)$$

$$\begin{tikzcd} \mathcal{O}_{\mathbb{C}^m, f(x)} \arrow[r] & \mathcal{O}_{\mathbb{C}^n, x} \arrow[d] \\ \mathcal{O}_{\mathbb{C}^m, f(x)}/\mathfrak{m} \arrow[r] \arrow[d, phantom, "\parallel"] & \mathcal{O}_{\mathbb{C}^n, x} \arrow[d] \\ \mathbb{C} \arrow[r, phantom, "="] & \mathbb{C} \end{tikzcd}$$

$$f^\#(g)(x) = f^\#(g)/\mathfrak{m}_x$$
$$\parallel$$
$$g(f(x)) = g/\mathfrak{m}_{f(x)}$$

$$\Rightarrow \underline{f^\#(g) = g \circ f}$$

Hence $f^\#$ is uniquely determined by f. in this example

3. Complex spaces

Let $U \subset \mathbb{C}^n$ be open. $A \subset U$ is an analytic set $:\Leftrightarrow$

$\forall x \in A \;\exists\; V(x)$ & $f_1, \dots, f_k \in \mathcal{O}(V)$ s.t.

$$A \cap V = V(f_1, \dots, f_k) := \{y \in V \mid f_i(y) = 0,\ i = 1, \dots, k\}$$

Define the full ideal sheaf $\mathcal{Id}_A \subset \mathcal{O}_U$ of A by

$$\mathcal{Id}_A(V) = \{f \in \mathcal{O}_U(V) \mid f|_A \equiv 0\} \quad \forall V \subset U \text{ open}$$

Theorem I.1 (Oka): $\mathcal{O}_{\mathbb{C}^n}$ is coherent (i.e. relation finite)

Theorem I.2 (Cartan): $\mathcal{Id}_A$ is coherent for any analytic set A

Corollary: $\mathcal{O}_A := (\mathcal{O}_{\mathbb{C}^n} / \mathcal{Id}_A)|_A$ is coherent ($\mathcal{Id}_{A,x} = \mathcal{O}_{\mathbb{C}^n,x}$ for $x \notin A$)

Note: • If $\varphi: \mathcal{F} \to \mathcal{G}$ and $\mathcal{F}, \mathcal{G}$ coherent $\Rightarrow$ $\operatorname{Ker}\varphi$, $\operatorname{Im}\varphi$, $\operatorname{Cok}\varphi$ coherent)
• $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ exact sequence of sheaves. If two are coherent then the third is coherent
(i.e. exact on stalks)

$(A, \mathcal{O}_A)$ is a ringed space,
It is an example of a complex models space $(X, \mathcal{O}_X)$:

$U \subset \mathbb{C}^n$ open $\quad f_1, \dots, f_k \in \mathcal{O}(U)$

$$\mathcal{J} := f_1 \mathcal{O}_U + \dots + f_k \mathcal{O}_U \subset \mathcal{O}_U \text{ ideal sheaf}$$

$$X := V(\mathcal{J}) := \{x \in U \mid f_i(x) = 0,\ i = 1 \dots k\}$$

$$\mathcal{O}_X := (\mathcal{O}_U / \mathcal{J})|_X$$

the ringed space $(X, \mathcal{O}_X)$ is called a complex model space

Note: $\mathcal{J} \subset \mathcal{Id}_X$ (for general complex spaces $\mathcal{J}$

Definition: A complex space $(X, \mathcal{O}_X)$ is a $\mathbb{C}$-analytic ringed space s.t.

- X is Hausdorff
- $\forall x \in X \;\exists\; U(x)$: $(U, \mathcal{O}_X|_U)$ is isom. to a complex model space (as $\mathbb{C}$-analytic ringed space)

$\mathcal{O}_X$ = structure sheaf of X (is coherent)

Theorem (Hilbert-Rückert Nullstellensatz)

Let X be a complex space $\mathcal{J} \subset \mathcal{O}_X$ a coherent ideal (sheaf)

$Y := V(\mathcal{J}) := \{x \in X \mid \mathcal{J}_x \neq \mathcal{O}_{X,x}\}$, $\mathcal{Id}_Y(U) := \{f \in \mathcal{O}_X(U) \mid Y \cap U \subset V(f)\}$

$$\Rightarrow \mathcal{Id}_Y = \sqrt{\mathcal{J}} \qquad ((\sqrt{\mathcal{J}})_x = \sqrt{\mathcal{J}_x})$$

Set $\operatorname{Nil}(\mathcal{O}_X) = \sqrt{0} \subset \mathcal{O}_X \qquad \mathcal{O}_{X^{red}} = \mathcal{O}_X / \operatorname{Nil}(X)$ is coherent (Cartan)

$X^{red} := (X, \mathcal{O}_{X^{red}})$ reduction of X is a complex space

b.w.

X is ~~reduced~~ $\Leftrightarrow$ $X = X^{red}$ i.e. $\mathcal{O}_{X,x}$ has no nilpotent elts $\forall x \in X$

$\Leftrightarrow$ $\forall x \in X$ $\exists\, \mathcal{U}(x)$
s.t. $(\mathcal{U}, \mathcal{O}_X|_{\mathcal{U}}) \cong (A, \mathcal{O}_A)$ for A analytic set in some open set of $\mathbb{C}^n$

$\Leftrightarrow$ X is locally isomorphic to the complex model space of an analytic set

§4 Normal complex spaces

Let $X = (X, \mathcal{O}_X)$ complex space

$\mathcal{O}_{X^{red}} := \mathcal{O}_X / Nil(\mathcal{O}_X)$ $\quad Nil(\mathcal{O}_X) := \sqrt{0} \subset \mathcal{O}_X$ is coherent by Cartan

$X^{red} = (X, \mathcal{O}_{X^{red}})$ <u>reduction of X</u>, is a complex space

X is <u>reduced</u> $:\Leftrightarrow X = X^{red}$ i.e. $\mathcal{O}_{X,x}$ has no nilpot. elt $\forall x \in X$, $\mathcal{O}_{X,x}$ reduced

$\Leftrightarrow$ X is locally isomorphic to the complex model space of an analytic set

X is <u>normal</u> $:\Leftrightarrow$ X is reduced and $\mathcal{O}_{X,x}$ is normal $\forall x \in X$

i.e. $\mathcal{O}_{X,x}$ is integrally closed in $Quot(\mathcal{O}_{X,x})$

X is <u>regular</u> or <u>smooth</u> $\Leftrightarrow$ $\mathcal{O}_{X,x}$ is a regular local ring $\forall x \in X$

$\Leftrightarrow$ X is complex manifold (Jacobian criterion)

$Sing(X) := \{x \in X \mid \mathcal{O}_{X,x} \text{ not regular}\}$

$NNor(X) := \{x \in X \mid \mathcal{O}_{X,x} \text{ not normal}\}$

$NRed(X) := \{x \in X \mid \mathcal{O}_{X,x} \text{ not reduced}\}$

<u>Theorem I.3</u> (1) $Sing(X)$, $NNor(X)$, $NRed(X)$ are closed analytic subsets of X with $\quad$ (*) b.w.

$$NRed(X) \subset NNor(X) \subset Sing(X).$$

(2) If $X = X^{red} \Rightarrow Sing(X)$ is nowhere dense in X.

<u>Def.</u> Let X be reduced. A <u>normalization</u> of X consist of a normal complex space $\bar{X}$ and a morphism $\nu: \bar{X} \to X$ st.

(i) ν is surjective and finite (i.e. ν is closed & $|\nu^{-1}(x)| < \infty$ $\forall x \in X$)

(ii) $\forall A \subset X$ analytic and nowhere dense

$\nu^{-1}(A)$ is nowhere dense in $\bar{X}$ and

$$\nu: \bar{X} \setminus \nu^{-1}(A) \xrightarrow{\cong} X \setminus A.$$

<u>Theorem I.4</u> Let X be reduced $\Rightarrow$ unique

(1) X admits a normalization (up to unique isomorphis)

(2) If $f: Z \to X$ is a surj. morph, Z normal $\Rightarrow$ f factors through the normalization.

$$\begin{tikzcd} Z \arrow[r, "\exists"] \arrow[dr, "f"'] & \bar{X} \arrow[d, "\nu"] \\ & X \end{tikzcd}$$

X compl. space

$A \subset X$ <u>analytic</u> $\Leftrightarrow$ $\exists$ coherent ideal sheaf $\mathcal{J} \subset \mathcal{O}_X$

s.t. $A = V(\mathcal{J}) = \{x \in X \mid \mathcal{J}_x \neq \mathcal{O}_{X,x}\}$

$= \operatorname{supp}(\mathcal{O}_X/\mathcal{J})$.

(analytic sets are closed)

Theorem I.5. Let X be reduced

(1) a) X normal $\Rightarrow \dim(\mathrm{Sing}(X),x) \leq \dim(X,x)\ \forall x \in \mathrm{Sing}(X)$

b) If X is Cohen-Macaulay (i.e. $\mathcal{O}_{X,x}$ is CM $\forall x \in X$) the converse is also true

(2) X is normal $\Leftrightarrow$ $\forall\ \mathcal{U} \subset X$ open, the restriction map

$$\mathcal{O}_X(\mathcal{U}) \overset{\sim}{\Rightarrow} \mathcal{O}_X(\mathcal{U} \setminus \mathrm{Sing}(X) \cap \mathcal{U}) \text{ is bij.}$$

$\dim(X,x) := \mathrm{Kr.}\dim \mathcal{O}_{X,x}$.

Theorem I.6 Let X be reduced. The following are equivalent

(1) X is irreducible (i.e. $X \neq A_1 \cup A_2$ with A_i proper analytic subsets)

(2) $X \setminus \mathrm{Sing}(X)$ is connected

Theorem I.7. Every reduced complex space X has a unique decomposition into irreducible components, i.e.

- $X = \bigcup_{i \in I} X_i$, $X_i \subset X$ irreduc. analytic subsets
- $\{X_i\}$ is locally finite
- $X_i \not\subset X_j\ \forall i \neq j$

Note 1) In fact if $X \setminus \mathrm{Sing}(X) = \dot{\bigcup} X_i'$ with X_i' the connected components

$\Rightarrow X_i = \overline{X_i'}$ (top. closure in X) are the irreduc. comp. of X.

2) A compact complex space has only finitely many irreducible components

5. Singularities

X, x pointed complex space

$f: X, x \to Y, y$ morph. which resp. dist. points

category of <u>complex space germs</u>

- objects : pointed complex spaces
- morphisms: equivalence classes of morph. $f: \mathcal{U}, x \to Y, y$

$$\mathcal{U} = \mathcal{U}(x), \quad g: V(x), x \to Y, y$$

$$f \sim g \iff \exists\, W(x) \subset \mathcal{U} \cap V : f|_W = g|_W$$

(X, x) = germ of X at x. A <u>singularity</u> is a complex space germ

A <u>morphism</u> $(f, f^\#): (X, x) \to (Y, y)$ determines a morph. of analyt. algebras

$$\text{by } f^\#_x : \mathcal{O}_{Y, f(x)} \to \mathcal{O}_{X, x}$$

<u>Prop.</u>: $\{\text{complex space germs}\}/_{\text{iso}} \longleftrightarrow \{\text{analytic algebra}\}/_{\text{iso}}$

$$(X, x) \longmapsto \mathcal{O}_{X,x}$$

$$(f, f^\#) \longmapsto f^\#_x$$

is an antiequivalence of categories

<u>Pf</u>: lets show the inverse functor.

$$\mathcal{O}_{X,x} \cong \mathbb{C}\{x_1 \ldots x_n\}/I \qquad (X, x) := (V(I), 0) \subset (\mathbb{C}^n, 0)$$

given $\varphi: \mathbb{C}\{\underline{x}\}/I \to \mathbb{C}\{y\}/J$

$$\begin{array}{ccc} \mathbb{C}\{\underline{x}\}/I & \xrightarrow{\varphi} & \mathbb{C}\{y_1 \ldots y_m\}/J \\ \uparrow & & \uparrow \\ \mathbb{C}\{x\} & \xrightarrow[\exists \tilde\varphi]{} & \mathbb{C}\{\underline{y}\} \end{array} \qquad \tilde\varphi(I) \subset J \qquad \tilde f_i := \tilde\varphi(x_i) \in \mathbb{C}\{y\}, \ \in \mathfrak{m}$$

$$\tilde f : (\mathbb{C}^m_y, 0) \to (\mathbb{C}^n_x, 0) \qquad \tilde f(y) := (\tilde f_1(y), \ldots, \tilde f_m(y))$$

$$\tilde f : V(J) \to V(I) \qquad \text{since } \tilde f(I) \subset J$$

let $g \in I$ $\qquad g \circ \tilde f(y) = \sum c_\alpha f_1^{\alpha_1}(y) \cdots f_m^{\alpha_m}(y)$

$\tilde\varphi(g) \in J$ $\qquad \underbrace{\underbrace{g \circ \tilde f}_{\tilde\varphi(g)}(y)}_{} = 0 \quad \forall y \in V(J)$, $\quad \tilde\varphi(g) \in J$

$$\Rightarrow g|\tilde f(V(J)) \equiv 0 \qquad \text{and } \tilde f^\# = \tilde\varphi$$

$$\tilde f^\#(x_i) = \tilde f_i$$

$\Rightarrow f = \tilde f|_{(X,x)}$

$f^\# =$

Notations: Let (X,x) be a singularity, $\mathcal{O}_{X,x} = \mathcal{O}_{\mathbb{C}^n,x}/I$ its local ring

(1) (X,x) is regular $\Leftrightarrow$ $\mathcal{O}_{X,x}$ is a regular local ring
($\Leftrightarrow$ (X,x) is the germ of a complex manifold)
otherwise (X,x) is singular

(2) (X,x) is an isolated singularity $:\Leftrightarrow$ $\exists\, U(x) \subset X$: $U \setminus \{x\}$ is regular

(3) (X,x) is a hypersurface singularity $\Leftrightarrow$ $\mathcal{O}_{X,x} \cong \mathcal{O}_{\mathbb{C}^n,0}/\langle f\rangle$

(4) (X,x) is normal $\Leftrightarrow$ $\mathcal{O}_{X,x}$ is normal, i.e. reduced and integrally closed in its total ring of fractions

(5) (X,x) is Cohen Macaulay (CM) $\Leftrightarrow$ $\mathcal{O}_{X,x}$ is C.M, i.e.
$\dim \mathcal{O}_{X,x} = \operatorname{depth} \mathcal{O}_{X,x}$

(6) $\dim(X,x) = \dim \mathcal{O}_{X,x}$ (Krull dim)
$\dim X = \sup_{x\in X} \{\dim(X,x)\}$

(7) Let $(X,x) \neq \emptyset$ be reduced. Then (X,x) is irreducible
(i.e. not the union of two proper complex subgerms)
$\Leftrightarrow$ $\mathcal{O}_{X,x}$ is a domain $\Leftrightarrow$ I is a prime ideal

(8) Let $I = \sqrt{I} = \bigcap_{i=1}^{r} \mathfrak{p}_i$ the prime decomposition of I.
$(X_i,x) = (V(\mathfrak{p}_i),x)$. Then $(X,x) = \bigcup (X_i,x)$ is the decomp. of (X,x) into irreducible components.
We have $\dim(X,x) = \max_{i=1\dots r} \dim(X_i,x)$.

Facts: $(X,\mathcal{O}_X)$ a complex space

- $\operatorname{Sing}(X) := \{x\in X \mid (X,x) \text{ singular}\} \subset X$ analytic set ($\Rightarrow$ closed)
- If X is reduced, then $\dim(\operatorname{Sing}(X),x) < \dim(X,x)$ $\forall x\in X$ and $\operatorname{Reg}(X) = X \setminus \operatorname{Sing}(X)$ is open and dense in X.
- If (X,x) is reduced then $\exists\, U(x)$ s.t. (U,y) is reduced $\forall y \in U$. (Since $\mathcal{N}il(X)$ is coherent!)

We need:

- decomp. of X into (loc. finite) irred comp.
- normal & normalization & univers. property
- the irreducible components are the closure of the connected components of $X \setminus \mathrm{Sing}(X)$.
- proper coherence & proper mapping theorem.
- multiplicity
- $H^1(X, \mathcal{F})$ Čech cocycle Beal.