# Surface singularities II : Resolution of singularities

Methods to study singularities $X$

- direct algebraic: equations of $X$.
- stratification: decompos. of $X$ in disj. smooth subspaces
- deformation: deform equations of $X$
- resolution: dominate $X$ by a smooth space without changing $X \setminus \mathrm{Sing}(X)$.

## II.1. Definition and existence

Def: $X$ reduced complex space. A resolution of $X$ is a morphism

$$\pi : \tilde{X} \to X \quad \text{s.t.}$$

- $\tilde{X}$ is smooth
- $\pi$ is surjective & proper (i.e. $\pi^{-1}(\text{compact})$ is compact)
- $\pi^{-1}(\mathrm{Sing}(X))$ is nowhere dense in $\tilde{X}$
- $\pi : \tilde{X} \setminus \pi^{-1}(\mathrm{Sing}(X)) \xrightarrow{\cong} X \setminus \mathrm{Sing}(X)$

Theorem II.1 (Hironaka 1964)

(1) Every red. complex space has a resolution

(2) A resolution can be obtained by a finite sequence of blowing up smooth subspaces s.t. $\pi^{-1}(\mathrm{Sing}(X))$ is a divisor in $\tilde{X}$ (i.e. locally a hypersurface)

Note: Every resolution factors through the normalization (since $\tilde{X}$ is normal), hence may assume $X$ normal.

Theorem II.2 (Zariski, Abhyankar)

Let $S$ be a reduced surface ($\dim_x S = 2 \ \forall x \in S$). Then $S$ can be resolved by a finite sequence of normalizations and blowing up singular points.

Exercise: Show that $S = \{x^2 - y^2 z^2 = 0\}$ cannot be resolved by only blowing up points.

## I.2 Blowing up points:

$M = \mathbb{C}^n \qquad \lambda : \mathbb{C}^n \setminus \{0\} \twoheadrightarrow \mathbb{P}^{n-1} = \mathbb{C}^n \setminus \{0\} / \mathbb{C}^*$ canonical

$\Gamma_\lambda := \{ (x, \ell) \in (\mathbb{C}^n \setminus \{0\}) \times \mathbb{P}^{n-1} \mid \underset{\text{i.e. } x \in \ell}{\lambda(x) = \ell} \}$ graph of $\lambda$

$M' := \overline{\Gamma_\lambda} \subset \mathbb{C}^n \times \mathbb{P}^{n-1}$ top. closure

$bl_0(M) := M' = \{ (x_1, \dots, x_n, z_1 : \dots : z_n) \in \mathbb{C}^n \times \mathbb{P}^{n-1} \mid x_i z_j = x_j z_i \ \forall i,j \}$

$\downarrow \pi$ 1st projection <u>blowing up $0 \in M$</u>

$\mathbb{C}^n$

<u>Facts</u>:

(1) $\pi^{-1}(0) = \{0\} \times \mathbb{P}^{n-1}$ ✓

(2) $\pi : M' \setminus \pi^{-1}(0) \xrightarrow{\cong} M \setminus \{0\} : \ \pi^{-1}(a) = (a, \overline{0a}) \in \mathbb{C}^n \times \mathbb{P}^{n-1}$

(3) $M'$ is a complex manifold, $\dim M' = n$, covered by $n$ charts $\mathcal{U}_i := \{ (x,z) \in M' \mid z_i \neq 0 \}$

$\cong \{ (x,z) \in \mathbb{C}^n \times \{ z_i \neq 0 \} \mid x_j = \frac{x_i}{z_i} z_j \}$

$\cong z_i = 1$

$(w_1, \dots, w_n) \overset{\varphi_i}{\longmapsto} (w_i w_1, \dots, \underset{i}{w_i}, \dots, w_i w_n) \xrightarrow{\varphi_i} \ = \{ ( x_i z_1, \dots, \underset{i}{x_i}, \dots, x_i z_n), z_1, \dots, \underset{i}{1}, \dots z_n ) \}$

$\mathbb{C}^n = \{ (w_1, \dots, w_n) \} \ \cong \{ (z_1, \dots, \overset{i}{x_i}, \dots z_n) \}$

$\varphi_j \circ \varphi_i^{-1} : \varphi_i(\mathcal{U}_i \cap \mathcal{U}_j) \to \varphi_j(\mathcal{U}_i \cap \mathcal{U}_j)$ is holomorphic

(4) $\pi$ is proper: let $K \subset M$ compact $\Rightarrow$

$\pi^{-1}(K) = \underbrace{K \times \mathbb{P}^{n-1}}_{\text{comp.}} \cap \underbrace{M'}_{\text{closed}}$ is compact

<u>Remark</u>: Let $M$ is an arbitrary complex $n$-dim manifold, $p \in M$. Choose local coord. around $p$ and define $M' = bl_p(M)$ w.r.t. these coordinates.

$M'$ is unique up to isomorphism.

$bl_p(M) \to M$ : "<u>blowing up point</u>", "<u>monoidal transformation</u>"

"<u>$\sigma$-process</u>"

## II.3 Strict and total transform

Let $f:(\mathbb{C}^n,0)\to(\mathbb{C},0)$ be holom $\quad M=U(0)\subset\mathbb{C}^n$

$(X,0)=V(f)$ hypersurface singularity

$$f=\sum_{d\geq m} f_d \quad f_d \text{ homog. of deg } d\,,\quad m=\mathrm{mult}(f),\ f_m\neq 0$$

$\pi: M'\to M,0$ blowing up $0$

$$\begin{array}{ccc} M' & \xrightarrow{\pi} & M,0 \\ & {\scriptstyle f\circ\pi}\searrow & \downarrow f \\ & & \mathbb{C},0 \end{array}$$

Write $f\circ\pi|_{U_i}$ in local coord. of $U_i$

$$f\circ\pi|_{U_i} := f\circ\pi\circ\varphi_i(w_1,\dots,w_n)=f(w_iw_1,\dots,w_i,\dots,w_iw_n)$$

$$= w_i^m\cdot\underbrace{\sum_{d\geq m} w_i^{d-m} f_d(w_1,\dots,1,\dots w_n)}_{f_i'(w_1,\dots,w_n)}$$

<u>Note</u> On $U_i\cap U_j$: $V(w_i)=V(w_j)$, $V(f_i')=V(f_j')$

$E:=$ the divisor on $M'$ defined by $w_i=0$ on $U_i$

<u>exceptional divisor</u> of $\pi$

$X':=$ divisor on $M'$ defined by $f_i'=0$ on $U_i$

<u>strict</u> (or <u>proper</u>) <u>transform</u> of $X$

$(\pi^*f)=mE+X'$ divisor defined by $f\circ\pi=w_i^m\cdot f_i'=0$ on $U_i$

<u>total transform</u>

$E+X'$: defined by $w_if_i'=0$ on $U_i$

<u>reduced total transform</u>

<u>Remark</u>:
- $E=\pi^{-1}(0)\cong\mathbb{P}^{n-1}$, $M'\setminus E\xrightarrow{\cong}M\setminus\{0\}$,
- $X'\setminus E\cap X'=\pi^{-1}(X\setminus\{0\})\xrightarrow{\cong}X\setminus\{0\}$
- $X'=\overline{\pi^{-1}(X\setminus\{0\})}$ top. closure in $M'$

## II.4 Embedded resolution of plane curve singularities



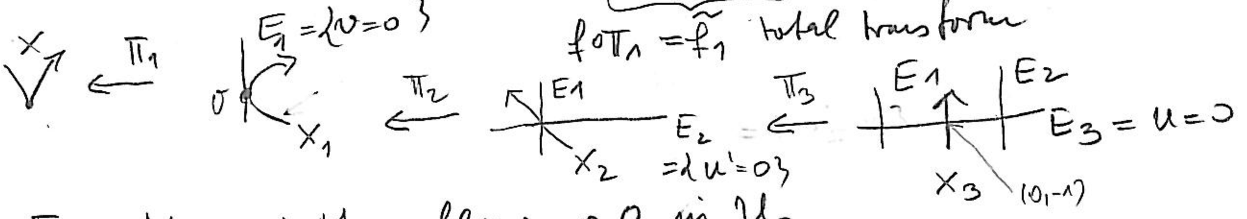
<u>Example</u> $f(x,y) = x^2 + y^3$ $\quad X = V(f) \subset M_0 = \mathbb{C}^2$ ✓

$\pi_1 : M_1 \to M_0$ blow up $0$

in $U_1$: $f \circ \pi_1(u,v) = f(u,uv) = u^2 \cdot \underbrace{(1+uv^3)}_{\text{strict transf. } f_1 \, (=f_1')}$ $\quad E = \{u=0\}$

$(X_1 = V(f_1)) \cap E = \emptyset$ not interesting

in $U_2$: $f \circ \pi(u,v) = f(uv,v) = \underbrace{v^2 \underbrace{(u^2+v)}_{f_1 \, (=f_2')}}_{f \circ \pi_1 = \tilde{f}_1 \text{ total transform}}$ $\quad X_1 \cap E = \{0\}$




$\pi_2 : M_2 \to M_1$ blows up $0$ in $U_2$

in $U_1$: $\tilde{f}_1 \circ \pi_2(u',v') = \tilde{f}_1(u', u'v') = \underbrace{u'^3 (v'^2 \underbrace{(u'+v')}_{f_2})}_{\tilde{f}_2}$, $\quad X_2 = V(f_2)$

(in $U_2$ not interesting)

$\pi_3 : M_3 \to M_1$ blow up $0$ in $U_1$

$U_1$: $\tilde{f}_2 \circ \pi_3(u,v) = \tilde{f}_2(u,uv) = \underbrace{u^6 (v^2 \underbrace{(1+v)}_{f_3 \text{ strict transform}})}_{\tilde{f}_3 \text{ total transform}}$

in $U_2$ analog

$(X_3 = V(f_3)) \cap (E_3 = \{u=0\}) = \{(0,-1)\}$

<u>Theorem</u> II.3 (Newton, Kronecker, M. Noether ...)

Let $(X,p) \subset (\mathbb{C}^2,p)$ be a reduced plane curve singularity $\Rightarrow$

$\exists$ a finite sequence of blowing up points

$\pi : M_N \xrightarrow{\pi_N} M_{N-1} \to \dots \to M_1 \xrightarrow{\pi_1} M_0 = U(p) \subset \mathbb{C}^2$ s.t.

(1) The total transform $(\pi^* f)$ $(\tilde{f}_N) =$ is a divisor with normal crossing (in local coord. $x^p y^q = 0$ in $M_N$)

(2) $\pi : X_N \to X$ is a resolution of $X$ $\quad$ ($X_N$ = strict transform)

<u>Idea of proof</u>: • blowing up does not increase mult$(f)$

- mult$(f)$ decreases after finitely many steps $\Rightarrow$
- mult$(f_n) = 1$ for some $n \Rightarrow f_n$ smooth, $V(f_n) \to V(f)$ resol.
- further blowing up get normal crossings of $(\tilde{f}_N)$

## 5. Resolution of normal surface singularities

Let $(S,p)$ be a normal surface singularity $\Rightarrow$ $\exists$ representative $S$:

- $S$ is normal, connected
- $S \setminus \{p\}$ is smooth

**Proposition II.4.** Let $\pi: X \to S$ be any resolution of $S$, $E := \pi^{-1}(p)_{red}$, $S$ non-smooth $\Rightarrow$

(i) $X$ is connected, $E$ is a compact connected curve

(ii) Let $E = \bigcup_{i=1}^{s} E_i$ be the decompos. of $E$ into irred. components $\Rightarrow$ $E_i$ is compact, the $E_i$ meet only in finitely many points

Proof. (i) $E$ is compact since $\pi$ is proper, any cpt space has only finitely many irred. components

We show that $E$ is connected:

Assume $E = K_1 \cup K_2$, $K_1 \cap K_2 = \emptyset$, $K_i \neq \emptyset$ a union of conn. comp. of $E$

$K_i$ compact, $X$ Hausdorff $\Rightarrow$

$\exists\, \mathcal{U}_i = \mathcal{U}_i(K_i) \subset X$ open $\mathcal{U}_i \cap \mathcal{U}_j = \emptyset$ & $\mathcal{U}_1 \cup \mathcal{U}_2$ neigh of $E$.

Claim: $\pi: X \to Y$ a proper, continuous map, $p \in Y$ $\Rightarrow$ $\{\pi^{-1}(V) \mid V = V(p)\}$ is a basis of the neighborhoods of $\pi^{-1}(p)$.

i.e. $\forall\, \mathcal{U} = \mathcal{U}(\pi^{-1}(p))$ $\exists\, V = V(p): \pi^{-1}(V) \subset \mathcal{U}$ (o.w.)

Proof Exercise in topology

$\Rightarrow \exists\, V = V(p): \pi^{-1}(V) \subset \mathcal{U}_1 \cup \mathcal{U}_2$ $\quad W_i := \mathcal{U}_i \cap \pi^{-1}(V)$, $W_1 \cap W_2 = \emptyset$

$$\underbrace{(W_1 - E) \cup (W_2 - E)}_{\text{disconnected}} = \pi^{-1}(V) \setminus E \xrightarrow{\cong} \underbrace{S \setminus \{p\}}_{\text{connected since } S \text{ irreducible (since } S \text{ normal)}}$$

$\lightning$

$\Rightarrow X$ connected & $E$ connected

(ii) $E \subset X$ nowhere dense $\Rightarrow \dim E < \dim X = 2$

a) If $\dim E = 0 \Rightarrow E = \{pt\}$ by (i) & $\pi: X - \{pt\} \xrightarrow{\cong} S - \{p\}$
$\Rightarrow \pi$ finite, $\pi$ is the normalization $\Rightarrow X \cong S \Rightarrow S$ smooth $\lightning$

b) $\Rightarrow E_i$ is a curve

$E_i \cap E_j \subset \operatorname{Sing}(E)$ $\quad \dim \operatorname{Sing}(E) = 0$ since $E$ reduced

$\Rightarrow \operatorname{Sing}(E)$ discrete & compact $\Rightarrow$ finite.

Proof of claim: Assume not $\Rightarrow$ for arbitrary small neighborhood $V_\nu$ (e.g. $B_{\frac{1}{\nu}}(p)$) $\pi^{-1}(V_\nu) \subset \mathcal{U} \Rightarrow \exists\, y_\nu \in V_\nu$, $\pi^{-1}(y_\nu) \not\subset \mathcal{U}$

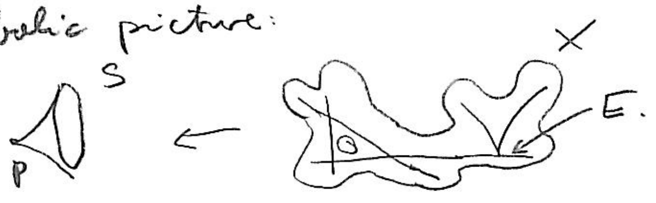
$y_\nu \longrightarrow p$, $\overline{V_\nu} \subset Y$ $\quad\quad \exists\, x_\nu \in \pi^{-1}(y_\nu)$, $x_\nu \notin \mathcal{U}$

$x_\nu \in \underset{\text{compact}}{\pi^{-1}(\overline{V_\nu})} =$ compact $\Rightarrow \{x_\nu\}$ has a limit point $x_0 \in \pi^{-1}(\overline{V_\nu}) \subset X$

$\Rightarrow x_0 \in \pi^{-1}(p)$ (otherwise $\pi(x_0) \neq p$ but $\pi(x_0) = \lim \pi(x_\nu) = p$ )

$\Rightarrow \exists\, \infty$-many $x_\nu$ in $\mathcal{U} \ni x_0$ $\quad$ ↯ $\quad$ to $x_\nu \notin \mathcal{U}$.

Symbolic picture:




**Theorem II.5.** Let $S$ be a normal surface singularity $\Rightarrow$ $\exists$ resolution $\pi : X \to S$ satisfying $E = \pi^{-1}(p)_{red} = \bigcup E_i$

(a) $E_i$ is a smooth curve

(b) $E_i \pitchfork E_j$ (intersect transversally)

(c) $x \in E$ lies on at most two components

(d) $\# E_i \cap E_j \leq 1 \quad \forall i \neq j$

I.e.  forbidden by (a)  forbidden by (b)

 forbidden by (c)  forbidden by (d)

If (a) – (c) hold: good (or transversal) resolution

If (a) – (d) hold: very good resolution

Proof: Choose any resolution of $S$. $E$ is a reduced curve by Th. II.4. By embedd. resol. of $E$ we can achieve (a) – (d).

Def: A resolution $\pi : X \to S$ is minimal $:\Leftrightarrow$ any other resolution $\pi' : X' \to S$ factors through $\pi$

$$\begin{tikzcd} X' \arrow[rr, dashed, "\exists \varphi"] \arrow[dr, "\pi'"'] & & X \arrow[dl, "\pi"] \\ & S & \end{tikzcd}$$

Note 1) $\varphi$ is uniquely determined ($X' \setminus \pi'^{-1}(p) \overset{\varphi}{\cong} X - \pi^{-1}(p)$ has a unique extension since $\pi^{-1}(p)$ is nowhere dense

2) A min. resolution is unique (up to unique iso) (proof similar)