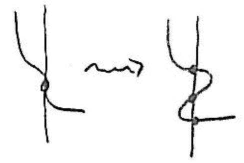
III. Intersection theory on surfaces

III.1 Geometric meaning of intersection numbers

Let M^n real n-dim C^∞ manifold, oriented.
$X^k, Y^{n-k} \subset M$ oriented submanifolds of complement. dim

(*) Deform X (or Y or both) a little to get X' s.t.
$X' \pitchfork Y$ transversal, i.e.
- X & Y meet in isolated points
- $T_x X' \oplus T_x Y = T_x M$



define
$$i(X',Y,x) := \begin{cases} +1 & T_xX'\oplus T_xY \text{ same orientation as } T_xM \\ -1 & \text{— " — opposite — " —} \end{cases}$$

Note: (*) is possible by Thom's transversality theorem for C^∞-maps, applied to $i: X \hookrightarrow M$.

Intersection product
$$X\cdot Y := X'\cdot Y := \sum_{x\in X'\cap Y} i(X',Y,x)$$ is finite if X or Y compact

Remark: 1) $X\cdot Y$ is independ. of "generic" choice of X'
2) $X\cdot X$ is defined
3) If M, X, Y are complex manifold $\Rightarrow$ they are oriented and $T_xY \oplus T_yY$ has same orient. as T_xM.
If $x\in X\cap Y$ is isolated point $\Rightarrow$
$$i(X,Y,x) > 0$$
However, a deformation X' as above is only a C^∞ but not a complex manifold $\Rightarrow i(X',Y,x)<0$ is possible
e.g. $X\cdot X < 0$ is possible
4) If S is a smooth (complex) surface, $X, Y \subset S$ curves, $p\in X\cap Y$ isolated point $\Rightarrow$
$$\boxed{i(X,Y,p) = \dim \mathbb{C}\{x,y\}/\langle f,g\rangle}$$
if x,y are loc. coord. at p and f,g local eq. of X,Y

III.2 Intersection numbers in homology

Let $\pi: X, E \to S, p$ be a resolution of a normal surface singularity

$\quad\;\; \cup \qquad\quad \cup$

$\quad\;\; C' \;\to\; C$ plane curve singularity

strict transform

$\Rightarrow$ $E + C'$ = reduced total transform

E compact, C' not compact

We need to compute $E \cdot C'$ in X (not compact)

Let M^{2n} be an oriented C^∞-mfd (e.g. a complex manifold)

- If M is compact we have the Poincaré-isomorphism

$$\Delta: H_{2n-k}(M,\mathbb{Z}) \xrightarrow{\cong} H^k(M,\mathbb{Z}) \xrightarrow{\not\cong} \mathrm{Hom}(H_k(M),\mathbb{Z})$$

Hom kills torsion! ($\cong$ mod torsion)

and define the intersection product on homology

$$H_{2n-k}(M) \times H_k(M) \to \mathbb{Z}$$

$$a \circ b := \Delta(a)(b).$$

- If M not compact: need Borel-Moore homology

$H_k^{BM}(M,\mathbb{Z})$ is defined by "infinite chains of singular simplices which are locally finite".

Note: Elements in H_k are represented by finite chains, hence chains have compact support (but not in H_k^{BM}).

$\exists$ Poincaré-isomorphism

$$\Delta: H^{BM}_{2n-k}(M,\mathbb{Z}) \xrightarrow{\cong} H^k(M,\mathbb{Z})$$

$H_*^{BM}(M) = H_*(M)$ if M is compact

and natural maps

$$\beta: H_{2n-k}(M) \to H^{BM}_{2n-k}(M)$$

Intersection product in homology

$$H_{2n-k}(M) \times H_k(M) \to \mathbb{Z}$$

$$a \circ b := \Delta(\beta(a))(b)$$

Important

M oriented $\Rightarrow \exists\ [M] \in H^{BM}_{2n}(M,\mathbb{Z}) \underset{\cong\mathbb{Z}}{=} \mathbb{Z}\cdot[M]$ $\quad [M]$ = fundamental class

e.g. $H^{BM}_{2n}(\mathbb{C}^n,\mathbb{Z}) = \mathbb{Z}$ while $H_{2n}(\mathbb{C}^n) = 0$.

III.3 Computation of intersection numbers

Let M be a smooth complex surface. Let

$$D = \sum n_i D_i \qquad n_i \in \mathbb{Z},\ D_i \subset M \text{ irred. curve (prime divisor)}$$

be a divisor on M, not nec. compact

D defines a homology class

$$[D] := \sum n_i [D_i] \in H_2^{BM}(M, \mathbb{Z}), \qquad \underset{\text{fund. class}}{[D_i]} \in H_2^{BM}(D_i) \longrightarrow H_2^{BM}(M)$$

Intersection product for divisors

$$\mathrm{Div}(M) \times \mathrm{Div}(M) \longrightarrow \mathbb{Z}$$

$$D \cdot D' := \Delta([D])([D']), \quad \text{symmetric, bilinear}$$

Remark: Let D be given by a cover $\{\mathcal{U}_i\}$ of M and meromorphic functions $f_i \in \mathcal{M}^*(\mathcal{U}_i)$ s.t. $f_i = g_{ij} f_j$ $\forall i,j$, $g_{ij} \in \mathcal{O}_M^*(\mathcal{U}_i)$

$\{g_{ij}\}$ are the transition functions of an invertible sheaf (= vb. of rk 1) $\mathcal{O}_M(D) \in H^1(M, \mathcal{O}_M^*)$.

Recall that

$$H^1(\mathcal{O}_M^*) \xrightarrow{c_1} H^2(M, \mathbb{Z}) \qquad (\text{from } 0 \to \mathbb{Z} \to \mathcal{O}_M \xrightarrow{\exp(2\pi i\, -)} \mathcal{O}_M^* \to 0)$$

$c_1(D) := c_1(\mathcal{O}_M(D))$ first chern class of D

Theorem (Borel, Haefliger): If $D > 0$ ($n_i \geq 0$ $\exists\, n_i > 0$)

$$\Rightarrow \qquad \Delta([D]) = c_1(D).$$

Corollary: Let $f \in \Gamma(M, \mathcal{O}_M)$ not const $\Rightarrow$

$$(f) \cdot [D] = 0 \qquad \forall\, D \in \mathrm{Div}(M)$$

Proof: (f) is the divisor of the global hol. section hence $\mathcal{O}_M((f)) \cong \mathcal{O}_M \Rightarrow c_1((f)) = 0$ & $c_1((f))(D) = 0$ $\forall D$

Lemma: Let $M = bl_0 \mathbb{C}^2 \xrightarrow{\pi} \mathbb{C}^2$ point blow up, $E = \pi^{-1}(0)$

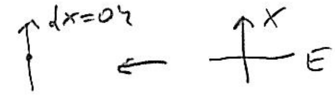
$$\Rightarrow \qquad E \cdot E = -1$$

Proof. Choose coordinate fct $x : \mathbb{C}^2 \to \mathbb{C}$, $f := x \circ \pi : M \to \mathbb{C}^2 \to \mathbb{C}$.

$X :=$ strict transform of $\{x = 0\}$ $\Rightarrow$

$$(f) = X + E \qquad \text{since mult}(x) = 1.$$

$$\Rightarrow 0 = (f) \cdot E = (X + E)E = \underbrace{X \cdot E}_{1} + E^2$$



III.4 Intersection matrix of a resolution

Let $\pi : X \to S$ be a resolution of a normal surface sing. S, p

$E = E_1 + \dots + E_s$ the except. divisor $\quad (= \pi^{-1}(p)_{red})$

$(E_i \cdot E_j)_{1 \le i,j \le s}$ <u>intersection matrix</u>

<u>Theorem</u>: The intersection matrix is symmetric and negative definite.

<u>Proof</u>: (1) Choose $f : S, p \to \mathbb{C}, 0$, $f \not\equiv 0$, $C = V(f) \subset S$ a curve

$$(\pi^* f) = \underbrace{\sum m_i E_i}_{\text{except. cycle}} + \underbrace{C'}_{\text{strict transf}} \qquad m_i > 0 \quad \left(\text{since } (f \circ \pi)^{-1}(0) \text{ is a curve in } X\right)$$

(2) Since $C' = \overline{\pi^{-1}(C \setminus \{p\})} \Rightarrow C' \cap E \neq \emptyset \Rightarrow C' \cdot E_j > 0$ some j_0

(3) It suffices to show that $(\alpha_{ij}) := (m_i E_i \cdot m_j E_j)$ is pos. def

(4) $$0 = (\pi^* f) \cdot m_j E_j = \left(\sum_i m_i E_i\right) m_j E_j + \underbrace{C' m_j E_j}_{\geq 0 \text{ and } \underline{> 0 \text{ some } j}}$$

$$\geq \sum_i \alpha_{ij} \quad \text{and} > \text{for some } j_0$$

(5) Look at the quadratic form $(x_1, \dots, x_s)(\alpha_{ij})\begin{pmatrix} x_1 \\ \vdots \\ x_s \end{pmatrix}$ on $\mathbb{R}^s$.

$$\sum_{i,j} \alpha_{ij} x_i x_j = \sum_i \alpha_{ii} x_i^2 + 2 \sum_{i<j} \alpha_{ij} x_i x_j \qquad (\alpha_{ij}) \text{ symmetric}$$

$$= \sum_j \underbrace{\left(\sum_i \alpha_{ij}\right)}_{\leq 0} x_j^2 - \sum_{i<j} \underbrace{\alpha_{ij}}_{\geq 0} (x_i - x_j)^2 \leq 0 \qquad \alpha_{ij} \text{ symmetric}$$

$\Rightarrow (\alpha_{ij})$ neg. semidefinite

(6) Set $(5) = 0 \Rightarrow x_{j_0} = 0$ by (4)

(7) Since E connected $\Rightarrow \exists\, j_1 \neq j_0 : E_{j_1} \cap E_{j_0} \neq \emptyset$

$\Rightarrow \alpha_{j_0 j_1} > 0 \quad \Rightarrow x_{j_1} = 0$

$\exists\, j_2 : E_{j_2} \cap (E_{j_1} \cup E_{j_0}) \neq 0 \Rightarrow \alpha_{j_0 j_2}$ or $\alpha_{j_1 j_2} > 0$

$\Rightarrow x_{j_2} = 0$ etc.

III.5 Existence of a minimal resolution

A resolution $\pi: X \to S$ of a normal surface singularity (S,p) is minimal if every other resolution factors through it.

It is unique up to unique isom.

Theorem (Castelnuovo) Let X be any smooth surface and $C \subset S$ a compact curve $\Rightarrow$

$$\left.\begin{array}{l} \exists \text{ a smooth surface } Y,\ y \in Y \ \& \\ \text{a point blow up } \pi: X \to Y \text{ such} \\ \text{that } C = \pi^{-1}(y) \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} C \cong \mathbb{P}^1 \\ \underbrace{C \cdot C = -1} \\ C = -1 \text{ curve} \end{array}\right.$$

i.e. C can be blown down to a smooth manifold $Y \Leftrightarrow C$ is a -1 curve.

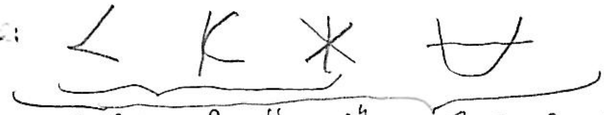
Theorem (Hopf) Let $\varphi: X \to Y$ be a proper holom map between smooth surfaces. Let $Z \subsetneq Y$ be a compact subvariety ($\dim Z < 2$) s.t.
$\varphi: X \setminus \varphi^{-1}(Z) \xrightarrow{\cong} Y \setminus Z$, $\Rightarrow$
φ is the composition of finitely many point blow ups.

Theorem (1) A resolution $\pi: X \to S$, of a normal surface singularity (S,p) is minimal $\Leftrightarrow \pi^{-1}(p)$ contains no -1 curves.

(2) The minimial resolution of a normal surface singularity exists. It is obtained from an arbitrary resolution by successively blowing down -1 curves.

(3) There exists a minimal (very) good resolution of S, i.e. every (very) good resolution of S factors through it.

For E are forbidden:



forbidden for "good" forbid. for very good

Idea of the proof (1) "$\Rightarrow$" from Casteln. & Hopf

"$\Leftarrow$" The main steps (which are of independent interest) are:

Theorem: Let π, π' be two resolutions of S. $\Rightarrow$ $\exists$ resolution π'' of S making the following diagram commute

$$\begin{tikzcd} & X'' \arrow[dl, "\theta'"'] \arrow[dr, "\theta"] \arrow[dd, "\pi''"] & \\ X' \arrow[dr, "\pi'"'] & & X \arrow[dl, "\pi"] \\ & S & \end{tikzcd}$$

with θ, θ' iterated point blowing ups.

follows from Hopf (using a graph construction)

Proposition: Let $C \subset X$ be an irred compact curve in a smooth surface X, $\pi: X' \to X$ blow up of $p \in C$, $C' \subset X'$ the strict transform of C. $\Rightarrow$

(1) If C smooth $\Rightarrow (C')^2 = C^2 - 1$

(2) If C is exceptional not -1 $\Rightarrow$ C' is not -1 curve