

IV. 1 Rational singularities

Let C a compact smooth complex curve \Rightarrow

topological: $C \cong \text{a wavy loop} \cong S^2 \cup \# g \text{ handles}$

$$g(C) = \text{genus}(C)$$



holomorphic: $g(C) = h^1(\mathcal{O}_C) = \dim_{\mathbb{C}} H^1(C, \mathcal{O}_C)$

If C is singular (but reduced), $\pi: \tilde{C} \rightarrow C$ a resolution

$$g(C) := g(\tilde{C}) \quad \text{"geometric genus"}$$

C is called rational $\Leftrightarrow g(C) = 0 \Leftrightarrow \tilde{C} \cong \mathbb{P}^1$

Let $f: X \rightarrow Y$ any morphism of complex spaces; F a sheaf on X . Define the higher direct image sheaves $R^i f_*(F)$ the sheaf associated to the presheaf

$$Y \ni V \mapsto H^i(f^{-1}(V), F) \quad i \geq 0.$$

$$\text{Since } H^0(-, F) = \Gamma(-, F) \Rightarrow R^0 f_* F = f_* F$$

Theorem (Frument)

If f is proper and F coherent $\Rightarrow R^i f_* F$ is coherent

We want to define a genus for a normal surface singularity (S, p) .

Let $\pi: X \rightarrow S, p$ be a resolution of S .

• $(R^i \pi_* \mathcal{O}_X)_q = H^i(\pi^{-1}(q), \mathcal{O}_{\tilde{X}}) \quad \forall q \in S \exists \text{ basis of neighborhoods } V = V(q) \text{ n.t. } \circ \text{ holds}$

$$q \neq p \quad = 0 \quad \text{since } \pi^{-1}(q) \xrightarrow{\sim} V$$

• $\Rightarrow (R^i \pi_* \mathcal{O}_X)_p = H^i(\pi^{-1}(p), \mathcal{O}_{\pi^{-1}(p)})$ is finite dim \mathbb{C} -vector space
 (since $R^i \pi_* \mathcal{O}_X$ is concentrated on p and coherent)

(2)

Def: 1) $p_g(S, p) := \dim_{\mathbb{C}} (R^1 \pi_* \mathcal{O}_X)_p = H^1(X, \mathcal{O}_X)$ (for small Stein neighbourhood of p)
 is called the geometric genus of (S, p) (analogous to curves)
 2) (S, p) is rational $\Leftrightarrow p_g(S, p) = 0$.

Note: $p_g(S, p)$ is independent of the resolution:

- any two resolutions $X \rightarrow S$, $X' \rightarrow S$ are dominated by a resolution $X'' \rightarrow S$ obtained from X and from X' by blowing up points
- $H^1(X, \mathcal{O}_X)$ does not change when blowing up a point.

Theorem (Artin) Let S be a rational singularity \Rightarrow

- Every resolution of a rational singularity is very good
 i.e. (a) E_i are smooth (b) $E_i \cdot E_j \in \{0, 1\}$ (c) $E_i \cap E_j \cap E_k = \emptyset$
 $i \neq j$ $i \neq j \neq k$

- Moreover, E_i is rational & the system of curves has no cycles (i.e. the dual graph is a tree)

(another justification of the name "rational")

IV.2 Rational double points generic

Let $(S, p) \subset (\mathbb{C}^n, p)$ and $\mathbb{C}^{n(n-2)}$ -dim linear space through p
 given by linear functions $l_1, \dots, l_{n(n-2)}$

$$\text{mult}(S, p) := \dim_{\mathbb{C}} \mathcal{O}_{S, p} / \langle l_1, \dots, l_{n(n-2)} \rangle$$

multiplicity of (S, p) (= Hilbert-Samuel mult of $\mathcal{O}_{S, p}$)

$$\text{e.dim}(S, p) = \dim_{\mathbb{C}} (\mathcal{M}_{S, p} / \mathcal{M}_{S, p}^2)$$

embedding dimension

Theorem (Artin) For a normal surface singularity (S, p)

$$\text{e.dim}(S, p) = \text{mult}(S, p) + 1$$

Corollary \checkmark ^{The rational singularity} (S, p) is (isomorphic to) a hypersurface singularity in (\mathbb{C}^3, p) $\Leftrightarrow \text{mult}(S, p) = 2$
 We say then (S, p) is a rational double point

One of the highlights in singularity theory is the
following Theorem (3)

Theorem (Artin, Briskorn, Du Val)

Let $(S, p) \subset (\mathbb{C}^3, p)$ be an isolated hypersurface singularity. The following are equivalent:

- (1) (S, p) is rational (i.e. a rational double point)
- (2) The minimal resolution of (S, p) is very good, the E_i are rational with $E_i^2 = -2$ and the dual graph of the resolution is one of the following

$$A_m : \begin{array}{ccccccc} & -2 & -2 & -2 & \cdots & \cdots & \\ \textcircled{o} & \textcircled{o} & \textcircled{o} & & & & m \geq 1 \\ E_1 & E_2 & & \cdots & & E_n & \end{array}$$

$$D_m : \begin{array}{ccccc} & -2 & & & \\ \textcircled{o} & \textcircled{o} & \textcircled{o} & \cdots & E_m \\ E_1 & E_3 & & & \\ & E_2 & -2 & & \\ & & & \cdots & \end{array} \quad m \geq 4$$

$$E_6 : \begin{array}{ccccc} & -2 & -2 & -2 & \\ \textcircled{o} & \textcircled{o} & \textcircled{o} & \textcircled{o} & \\ & & & & -2 \\ & & & & \textcircled{o} \\ & & & & -2 \end{array} \quad \text{cycle}$$

$$E_7 : \begin{array}{ccccc} & -2 & & & -2 \\ \textcircled{o} & \textcircled{o} & \textcircled{o} & \textcircled{o} & \\ & & & & -2 \\ & & & & \textcircled{o} \\ & & & & -2 \end{array}$$

$$E_8 : \begin{array}{ccccc} & -2 & & & -2 \\ \textcircled{o} & \textcircled{o} & \textcircled{o} & \textcircled{o} & \\ & & & & -2 \\ & & & & \textcircled{o} \\ & & & & -2 \end{array}$$

- (3) (S, p) is absolutely isolated, i.e. can be resolved by blowing up points (no normalization needed)

(absolute isolation holds for all rational singularities)

IV. 3 Quotient singularity

(4)

Let $G \subset GL(2, \mathbb{C})$ be a finite subgroup

G acts on \mathbb{C}^2 via matrix multiplication $(x, y) \cdot g$.

- via $\mathbb{C}[x, y]$ via $(g \cdot f)(x, y) = f((x, y) \cdot g^{-1})$

$$\mathbb{C}[x, y]^G = \{f \in \mathbb{C}[x, y] \mid g \cdot f = f \quad \forall g \in G\} \subset \mathbb{C}[x, y]$$

\Rightarrow (by E. Noether) a finitely generated \mathbb{C} -algebra

\exists generators $X_1, \dots, X_n \in \mathbb{C}[x, y]^G$, $X_i(0) = 0$, and

relations $f_i \in \mathbb{C}[x_1, \dots, x_n]$, $i=1, \dots, k$ s.t. $f_i(X_1, \dots, X_n) = 0$. $\forall i$.

\Rightarrow the canonical map

$$\mathbb{C}[x_1, \dots, x_n]/\langle f_1, \dots, f_k \rangle \xrightarrow{\cong} \mathbb{C}[x, y]^G$$

Hence we get a bijection

$$\mathbb{C}^2/G \xrightarrow{\text{bij}} X := \{x \in \mathbb{C}^n \mid f_1(x) = \dots = f_k(x) = 0\}$$

$$(x, y) \mapsto (X_1(x, y), \dots, X_n(x, y))$$

This makes \mathbb{C}^2/G an analytic (even algebraic) subset of \mathbb{C}^n and the germ

$$(\mathbb{C}^2/G, 0) = (X, 0)$$

is called a quotient singularity

Facts

- $(\mathbb{C}^2/G, 0)$ is a normal surface singularity
- $(\mathbb{C}^2/G, 0) \cong (\mathbb{C}^2/G', 0)$ for some $G' \subset GL(2, \mathbb{C})$, without pseudo-reflections (i.e. g is conjugate to $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ $2 \neq 0$)
- $(\mathbb{C}^2/G) \cong (\mathbb{C}^2/G')$ with G, G' small (Pruill)
 $\Rightarrow G$ and G' are conjugated in $GL(2, \mathbb{C})$
- (\mathbb{C}^2/G) is a hypersurface $\Leftrightarrow G \subset SL(2, \mathbb{C})$
 $(G$ small)
- Quotient singularities $(\mathbb{C}^2/G, 0)$ are rational surface singularities (Brieskorn)

Theorem: Let (S, p) be a normal surface singularity. Then equivalent (5)

(1) (S, p) is a rational double point.

(2) $(S, p) \cong (\mathbb{C}^2/G, 0)$ with $G \subset SL(2, \mathbb{C})$.

Du Val singularity

(3) $(S, p) \cong (V(f), 0)$ with f one of the following polynomials

$$A_n : x^{n+1} + y^2 + z^2 \quad n \geq 1$$

$$D_n : y(x^2 + y^{n-2}) + z^2 \quad n \geq 4$$

$$E_6 : x^3 + y^4 + z^2$$

$$E_7 : x(x^2 + y^3) + z^2$$

$$E_8 : x^3 + y^5 + z^2$$

(4) (S, p) is a hypersurface singularity and has finite Cohen Macaulay type (i.e. \exists only finitely many isoclasses of indecompos.

$\mathcal{O}_{S,p}$ - MCM)

(Buchweitz, G., Schreyer, J.
Kuiper)

Classification of finite subgroups of $SL(2, \mathbb{C})$ (F. Klein 1884)

Symbols $\mathcal{O}_{k+1}^{\text{Group}}$: Cyclic group of order $k+1$: $\mathcal{O}_{k+1} \cong \langle \begin{pmatrix} e & 0 \\ 0 & e^{-1} \end{pmatrix} \rangle \quad \mathfrak{e} = e^{2\pi i / k+1}$

D_k \mathcal{D}_{k-2} : Binary dihedral group of order $4(k-2)$
 $\langle \sigma, \tau \rangle \quad \sigma = \begin{pmatrix} 0 & 1 \\ 0 & \mathfrak{e}^{-1} \end{pmatrix}, \tau = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \mathfrak{e} = e^{2\pi i / k-2}$

E_6 T : Binary tetrahedral group, order = 24

E_7 O : Binary octahedral group : order 48

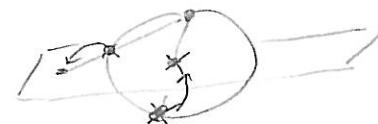
E_8 I : Binary icosahedral group : order 120

This classification is related to the classification
of Platonic solids (6)

- $G \subset \mathrm{SL}(2, \mathbb{C})$ finite is conjugated to $G' \subset \mathrm{SL}(2, \mathbb{C}) \quad \{A \mid A^{-1} = \bar{A}^t\}$

$$\mathrm{SU}(2, \mathbb{C}) \rightarrow \mathrm{PSU}(2, \mathbb{C}) = \mathrm{SU}(2, \mathbb{C}) / \{\pm 1\}$$

acts on $\mathbb{P}^1(\mathbb{C}) = \mathbb{C}^2 \setminus \{0\} / \mathbb{C}^*$
 S^2
 12 stereograph Proj.



via stereograph proj. The action of $\mathrm{PSU}(2, \mathbb{C})$ on $\mathbb{P}^1(\mathbb{C})$
 goes to the action of $\mathrm{SO}(3, \mathbb{R})$ on S^2 .

Hence $\{\text{finite subgroups of } \mathrm{PSU}(2, \mathbb{C})\} \leftrightarrow \{\text{finite subgroups of } \mathrm{SO}(3, \mathbb{R})\}$

\uparrow
 binary group in $\mathrm{SU}(2, \mathbb{C})$

C_n : cycle group: rotation group of regular n -vertex

D_n : symmetry group of dodecahedron

T : symmetry group of tetrahedron

O : octahedron

I : icosahedron.