Summary of the main achievements of Nicola Gigli

Gigli's major contributions are in three fields of analysis:

- Theory and applications of Gradient Flows,
- Structure of the Wasserstein space,
- Heat flow in non-smooth setting.

Theory and applications of Gradient Flows. This has been the theme on which Gigli spent most of his studies during the PhD. The results that I describe below are mostly contained in the first part of the book that Gigli, myself, and Giuseppe Savaré wrote on the topic: Gradient flows in metric spaces and in the Wasserstein space of probability measures (Birkhäuser), [2].
For a smooth functional $F : \mathbb{R}^d \to \mathbb{R}$ a gradient flow starting from $x^0 \in \mathbb{R}^d$ is a curve $[0, \infty) \ni t \mapsto x_t$ solution of

$$\begin{cases} x_t' &= -\nabla F(x_t), \qquad \forall t \geq 0 \\ x_0 &= x^0, \end{cases} \tag{1}$$

and the basic theory of ODE's ensures existence and uniqueness.
Shortly said, when studying the gradient flow problem in a metric setting, one has to do three things:

- find a metric analogous of the definition of gradient flow given by the above system,
- understand under which assumptions one gets existence,
- understand under which assumptions one gets uniqueness.

To define a metric analogous of (1) one recalls the definition of *metric speed* $|x_t'|$ of an absolutely continuous curve (x_t) in a metric space (X, d)

$$|x_t'| := \lim_{h \to 0} \frac{d(x_{t+h}, x_t)}{|h|},$$

and the definition of *slope* $|\nabla F|(x)$ of a functional $F : X \to \mathbb{R} \cup \{+\infty\}$, valid for any $x \in \{F < \infty\}$

$$|\nabla F|(x) := \overline{\lim_{y \to x}} \frac{\Big(F(x) - F(y)\Big)^+}{d(x, y)} = \max\left\{0, \overline{\lim_{y \to x}} \frac{F(x) - F(y)}{d(x, y)}\right\}.$$

It turns out that there are essentially 3 different generalizations of (1) to an abstract metric setting:

1) <u>Energy Dissipation Inequality</u> formulation:
We say that a locally absolutely continuous curve (x_t) is a gradient flow of the functional $F : X \to \mathbb{R} \cup \{+\infty\}$ starting from $x^0 \in \{F < \infty\}$ in the EDI formulation provided $\lim_{t \downarrow 0} x_t = x^0$ and for a.e. $t < s$ it holds

$$F(x_t) \geq F(x_s) + \frac{1}{2} \int_t^s |x_r'|^2 dr + \frac{1}{2} \int_t^s |\nabla F|^2(x_r) dr.$$

2) Energy Dissipation Equality formulation:
We say that a locally absolutely continuous curve (x_t) is a gradient flow of the functional $F : X \to \mathbb{R} \cup \{+\infty\}$ starting from $x^0 \in \{F < \infty\}$ the EDE formulation provided $\lim_{t \downarrow 0} x_t = x^0$ and for every $0 \leq t < s$ it holds

$$F(x_t) = F(x_s) + \frac{1}{2} \int_t^s |x_r'|^2 dr + \frac{1}{2} \int_t^s |\nabla F|^2(x_r) dr.$$

3) Evolution Variational Inequality formulation:
We say that a locally absolutely continuous curve (x_t) is a gradient flow of the functional $F : X \to \mathbb{R} \cup \{+\infty\}$ starting from $x^0 \in X$ the EVI formulation relative to $\lambda \in \mathbb{R}$ (EVI$_\lambda$ in short), provided $\lim_{t \downarrow 0} x_t = x^0$ and for any $t > 0$ it holds:

$$\lim_{h \downarrow 0} \frac{d^2(x_{t+h}, y)}{h} + F(x_t) + \frac{\lambda}{2} d^2(x_t, y) \leq F(y), \qquad \forall y \in X.$$

It is easy to see that for smooth functionals on $\mathbb{R}^d$ the EDI and EDE formulation of gradient flows are equivalent to the system (1), and that the EVI$_\lambda$ is equivalent to (1) provided the functional is $\lambda-$convex. Also, it is possible to see that the EVI$_\lambda$ is a definition stronger than the EDE which is in turn stronger than EDI.

While it may seem strange to have different definitions of gradient flow in a metric setting, in practice this is actually a useful tool because - obviously - the stronger hypotheses one has on the functional under study, the higher level of information one can get for the gradient flow. The existence results can be summarized as follows (for simplycity I will state them under compactness assumption much stronger than those actually needed):

EDI Assume that F is lower semicontinuous with compact sublevels, that $|\nabla F| : \{F < \infty\} \to \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous and that

$$\left.\begin{array}{rl} x_n & \to x, \\ \sup_n F(x_n) & < \infty, \\ \sup_n |\nabla F|(x_n) & < \infty, \end{array}\right\} \qquad \Rightarrow F(x_n) \to F(x).$$

Then for any $x^0 \in \{F < \infty\}$ there exists a gradient flow in the EDI sense starting from x^0.

EDE Assume that F is lower semicontinuous with compact sublevels and $\lambda-$geodesically convex[1] for some $\lambda \in \mathbb{R}$. Then for any $x^0 \in \{F < \infty\}$ there exists a gradient flow in the EDE sense starting from x^0.

EVI Assume that F is lower semicontinuous with compact sublevels and $\lambda-$convex along a certain class of curves bigger than the class of geodesics (that we called generalized geodesics) for some $\lambda \in \mathbb{R}$. Then for any $x^0 \in \overline{\{F < \infty\}}$ there exists a gradient flow in the EVI$_\lambda$ sense starting from x^0.

[1] i.e. for any couple of $x,\ y \in X$ there exists a constant speed geodesic joining them along which F is $\lambda-$convex.

All the three existence results are proved via a time discretization argument: one fixes a time parameter $\tau > 0$ and recursively defines a sequence x_n^τ by putting $x_0^\tau := x^0$ and choosing x_{n+1}^τ among the minimizers of

$$x \quad \mapsto \quad F(x) + \frac{d^2(x, x_n^\tau)}{2\tau}.$$

Then one rescales in time the discrete solution found by defining

$$x^\tau(t) := x^\tau_{[t/\tau]},$$

where $[\cdot]$ denotes the integer part.
Finally, an argument introduced by de Giorgi shows that for the rescaled curve a discrete version of the EDE is fullfilled and gives enough compatness to pass to the limit.

Concerning uniqueness, it should be noticed that in general the λ-geodesic convexity hypothesis (and thus the EDE formulation) is not enough to get uniqueness of the gradient flow. In the abstract metric setting something more is needed. Uniqueness and $\lambda-$contraction are consequences of the EVI_λ formulation.

Structure of the Wasserstein space In [22] Otto showed that the space $\mathscr{P}_2(\mathbb{R}^d)$ of probability measures with finite second moment endowed with the Wasserstein distance W_2 closely resembles a Riemannian manifold.
Gigli gave significative improvements to the understanding of such structure. The firsts have been obtained in collaboration with myself and Savaré in the aforementioned book:

- characterization of all absolutely continuous curves (μ_t) in the Wasserstein space via the continuity equation
$$\frac{d}{dt}\mu_t + \nabla \cdot (v_t \mu_t) = 0.$$
- Study of the properties of the *tangent space* $\mathrm{Tan}_\mu(\mathscr{P}_2(\mathbb{R}^d))$ to a measure $\mu \in \mathscr{P}_2(\mathbb{R}^d)$ defined by
$$\mathrm{Tan}_\mu(\mathscr{P}_2(\mathbb{R}^d)) := \overline{\left\{\nabla\varphi : \varphi \in C_c^\infty(\mathbb{R}^d)\right\}}^{L^2(\mathbb{R}^d,\mathbb{R}^d;\mu)}. \tag{2}$$
In particular, proof of the fact that *optimal maps are tangent*, which is a key statement to enable the study of the properties of geodesically convex functionals on $\mathscr{P}_2(\mathbb{R}^d)$.
- Introduction of the concept of subdifferential $\partial^W E$ of a geodesically convex functional E on $\mathscr{P}_2(\mathbb{R}^d)$ and study of the consequences in terms of gradient flows. Here the point is the following: once one has the identification of absolutely continuous curves via the continuity equation, a definition of tangent space and a definition of subdifferential of a geodesically convex functional, he can define the gradient flow of the functional $E : \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R} \cup \{+\infty\}$ as an absolutely continuous curve (μ_t) satisfying
$$\begin{cases} \dfrac{d}{dt}\mu_t + \nabla \cdot (v_t\mu_t) &= 0, \\ \qquad\qquad\qquad v_t &\in -\partial^W E(\mu_t), \qquad a.e.\ t > 0. \end{cases}$$

The question is then whether such definition coincides with the purely metric one described before in terms of Energy Dissipation Equality. Under general assumption the answers is yes, and this is result is one of those which tells that the weak Riemannian structure of the space $(\mathscr{P}_2(\mathbb{R}^d), W_2)$ is not just a formal analogy, but actually provides deep informations on the geometry and the analysis over such space.

- Having identified the 'metric' and the 'Riemannian' gradient flows of geodesically convex functionals, the problem is to understand under which assumptions we have existence. It turns out that many of the functionals that naturally appear in the theory (i.e. functionals of internal, potential and interaction energy type) are convex along a sufficiently large set of generalized geodesics, so that it is possible to prove existence, uniqueness and contractivity of gradient flows via the EVI_λ approach.

After the conclusion of his PhD, and after the appearence the book [2], Gigli continued studying the Riemannian properties of the space $(\mathscr{P}_2(\mathbb{R}^d), W_2)$. Basically, there are two things that his studied helped to clear: the structure of the tangent space and the second order calculus.
In (2) I recalled the definition of tangent space to a certain measure. However, both for theoretical and practical reasons, this space is sometimes 'too small': for instance, if μ is a Dirac delta, the set $\mathrm{Tan}_\mu(\mathscr{P}_2(\mathbb{R}^d))$ reduces to $\mathbb{R}^d$ and thus cannot reasonably describe the complexity of the Wasserstein space in a neighborhood of μ. For some applications it is better to consider a more geometric tangent space defined as

$$\mathbf{Tan}_\mu(\mathscr{P}_2(\mathbb{R}^d)) := \overline{\Big\{ \text{constant speed geodesics starting from } \mu \Big\}}^{D_\mu},$$

where D_μ is the distance defined by

$$D_\mu\big((\mu_t^1), (\mu_t^2)\big) := \varlimsup_{t \downarrow 0} \frac{W_2(\mu_t^1, \mu_t^2)}{t}.$$

This space was already introduced in the book [2], but its relation to the one in (2) was not completely clear. The problem of understanding the structure of the tangent space was also raised by Villani in his monograph [27] as a question remained open on the structure of $(\mathscr{P}_2(\mathbb{R}^d), W_2)$. Gigli in [13] proved that $\mathrm{Tan}_\mu(\mathscr{P}_2(\mathbb{R}^d))$ always embeds isometrically in $\mathbf{Tan}_\mu(\mathscr{P}_2(\mathbb{R}^d))$ and that the following are equivalent:

- the embedding of $\mathrm{Tan}_\mu(\mathscr{P}_2(\mathbb{R}^d))$ into $\mathbf{Tan}_\mu(\mathscr{P}_2(\mathbb{R}^d))$ is surjective,

- the space $(\mathbf{Tan}_\mu(\mathscr{P}_2(\mathbb{R}^d)), D_\mu)$ is an Hilbert space,

- μ has the following property: for any $\nu \in \mathscr{P}_2(\mathbb{R}^d)$ there exists only one optimal plan from μ to ν and this plan is induced by a map.

For the second order calculus, the point is the following: thanks to Otto's work, one knows that $(\mathscr{P}_2(\mathbb{R}^d), W_2)$ is somehow similar to a Riemannian manifold, and with the work in [2] one knows that it is possible to develop a rigorous first order calculus, i.e. to speak about derivative of a curve and differential of a functional even in a nonsmooth setting. Then, it is natural to ask whether the natural second order Riemannian objects like covariant derivative, parallel transport and curvature tensor are well defined on $(\mathscr{P}_2(M), W_2)$ or not. Independently from Gigli's studies, Lott [17] gave a positive, although purely formal, answer to this question. Gigli's approach - developed independently - has been a bit different, as he tried from the beginning to avoid smoothness assumptions and to provide not only a formal description of second order objects, but also existence results. The first results in these direction are contained in his PhD thesis [10] (and published in a joint work with myself [1]) and can be summarized as follows:
<u>'Sufficiency'</u>. One says that an absolutely continuous curve (μ_t) on $[0, 1]$ is *regular* provided the vector fields (v_t) identified by

$$\begin{aligned} &\frac{d}{dt}\mu_t + \nabla \cdot (v_t \mu_t) = 0, \\ &\qquad v_t \in \mathrm{Tan}_{\mu_t}(\mathscr{P}_2(\mathbb{R}^d)), \qquad q.e.\ t > 0, \end{aligned} \tag{3}$$

satisfy

$$\int_0^1 \mathrm{Lip}(v_t) dt < \infty.$$

Then along regular curves there is a well defined notion of covariant derivative and there exists the (unique) parallel transport.
<u>'Necessity'</u>. There are absolutely continuous curves (μ_t) such that the vector fields (v_t) identified by (3) satisfy

$$\mathrm{Lip}(v_t) \cong \frac{1}{t},$$

such that the parallel transport along them does not exist (curves of this kind can be chosen to be geodesics).
Gigli then further investigated the topic in [12]. He derived explicit formulas for the calculus of covariant derivatives and used them to rigorously introduce the curvature tensor $\mathbf{R}$ on $(\mathscr{P}_2(M), W_2)$ as a, typically unbounded, operator on $[\mathrm{Tan}_\mu(\mathscr{P}_2(M))]^4$ (it coincides with the one previously calculated by Lott in the case of smooth data). Along the same lines, he provided an explicit formula for the differentiation of the exponential map on $(\mathscr{P}_2(M), W_2)$ and proved that, as expected, the differential of the exponential map is the unique solution of the Jacobi equation.

Finally, a result concerning the regularity of the Kantorovich potential on non compact-manifolds, obtained in collaboration with Figalli ([7]), has been recently selected as 'highlighted paper' by ESAIM COCV.

Heat flow on metric measure spaces In [18] and [26] Lott-Villani and Sturm independently introduced a notion of metric measure space with Ricci curvature bounded below: $(X, d, \boldsymbol{m})$ has Ricci curvature

$\geq K$ provided the relative entropy functional

$$\mathrm{Ent}(\mu) := \begin{cases} \displaystyle\int \rho \log(\rho) d\boldsymbol{m}, & \mu = \rho \boldsymbol{m}, \\ +\infty, & \text{otherwise}, \end{cases}$$

is $K-$geodesically convex in $(\mathscr{P}_2(X), W_2)$. The study of the properties of these spaces has been a very rich research area in the past 5 years. The typical question is: which of those properties of Riemannian manifolds with Ricci curvature bounded below can be replicated in this more general setting? In particular it is natural to ask whether there is a well defined notion of Heat flow (recall that a sufficient condition on a manifold in order to be sure that the Heat flow does not lose mass is to have Ricci curvature bounded below).

Recalling that on a Riemannian manifold the Heat flow coincides with the gradient flow of the relative entropy w.r.t. W_2, it is natural to *define* the Heat flow in the abstract setting as the gradient flow of the relative entropy w.r.t. W_2. However, a priori it is not at all clear that this a good definition, as it is not obvious that for a given initial datum there is unique gradient flow[2]. The contribution of Gigli (see [14]) has been to show that uniqueness, in this setting, is actually true, so that the Heat flow is well defined in this abstract situation. In the same paper he showed that there is stability of the Heat flow under measured-Gromov-Hausdorff convergence of the base spaces.

In a slightly different direction, in a more recent paper in collaboration with K.Kuwada and S.I.Ohta (see [15]) he showed that in an Alexandrov setting, the gradient flow of the relative entropy w.r.t. W_2 coincides with the gradient flow of the Dirichlet energy $\mathcal{E}(f) := \frac{1}{2}\int |\nabla f|^2$ w.r.t. L^2. This generalizes the fact, well known in a smooth setting, that both the gradient flows produce the Heat flow. What is new in their approach is that the proof is purely metric, while previous results on the topic passed from PDE analysis (that's why such dualism was unknown in an unsmooth setting). An immediate consequence of their result is the regularizing property of the Heat flow in the Alexandrov situation: the Heat kernel is Lipschitz.

Finally, in a joint work in progress with me and Savaré, we are going to use the EVI formulation of the heat flow (somehow a condition stronger than convexity of the entropy) as weak definition of nonnegative Ricci curvature. We prove good tensorization and stability properties for this concept, as well consistency results for the "Wasserstein calculus" provided by the Fisher information functional (the rate of dissipation of Entropy) and the "Cheeger calculus" [3]. The latter makes sense in this context thanks to a result by Lott-Villani, who provide Poincaré inequalities in this class of spaces.

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[2] in general, the geodesic convexity of a functional does not imply uniqueness of its gradient flow. Furthermore, recent studies of Sturm and Ohta showed that in spaces with non negative Ricci curvature, the distance W_2 may not decrease along the gradient flow of the entropy, so that it isn't possible to deduce uniqueness from the contractivity.

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