Oberwolfach Prize 2022
Laudatio for Vesselin Dimitrov

Jean-Benoît Bost

It is a great pleasure to present the work of Vesselin Dimitrov for which he has been awarded the Oberwolfach Prize, whose topic in 2022 is *Algebra and Number Theory.*

Vesselin Dimitrov has contributed to several fundamental advances in number theory, more specifically to the so-called Diophantine geometry, and this *laudatio* is to a large extent an apologia for Diophantine geometry.

Actually the work of Vesselin Dimitrov constitutes a wonderful illustration of some ironic observations by various great masters of number theory concerning this field.

In relation to his own arithmetic contributions, Charles Hermite did write: *Il n'y a pas de méthode en théorie des nombres*,[1] to emphasize the unexpected role, when investigating Diophantine problems, of very diverse areas of mathematics — for instance in his work, of analytic techniques, and of the geometry of group actions and of Euclidean lattices.

In a similar vein, Jean-Pierre Serre and Jean-Marc Fontaine liked to say jokingly that *number theory is a part of applied mathematics.* The rationale behind this provocative sentence is that, while the expression "applied mathematics" usually alludes to applications of mathematics to other scientific areas (astronomy, physics, chemistry, data analysis, computing machinery, etc.), there also exist "inner applications" of mathematics, namely applications of mathematics to (a priori unrelated) basic problems of mathematics. It is notably the case of number theory, which nowadays relies on a considerable corpus of algebraic geometry, homological and homotopical algebra, representation theory, analysis on manifolds, etc.

I would like to briefly describe three major results of Vesselin Dimitrov. Each of them may be stated in rather elementary terms, while its proof relies on some remarkably clever and original use of an impressive diversity of techniques, and therefore provides a striking illustration of the above views of Hermite, Serre, and Fontaine.

The first of these results appears in his joint work with Z. Gao and P. Habegger, *Uniformity in Mordell-Lang for curves.*

Let us consider a smooth projective curve C, of genus g, defined over some number field K, namely an extension of the field $\mathbb{Q}$ of rational numbers of finite degree $d := [K : \mathbb{Q}]$. To this curve is canonically associated its Jacobian, $\mathrm{Jac}(C)$, an Abelian variety of dimension g over K.

A basic finiteness results concerning the Diophantine geometry of curves over number fields is due to Weil, who proved in 1929 that the abelian group $\mathrm{Jac}(C)(K)$ of K-rational points of the algebraic group $\mathrm{Jac}(C)$ is finitely generated.[2]

In 1984, Faltings achieved a fantastic breakthrough by proving that, when $g \geq 2$, the set of K-rational points $C(K)$ of the curve is finite, proving a long standing conjecture of Mordell. Faltings' theorem was given an alternative proof in 1991 by Vojta, by means of new techniques of Diophantine approximation. By elaborating considerably on Vojta's approach to Faltings' theorem, Dimitrov, Gao, and Habegger have established a bound for the cardinality of $C(K)$ when $g \geq 2$, of the form:

$$|C(K)| \leq c(g, d)^{1+\mathrm{rk}\,\mathrm{Jac}(C)(K)}, \tag{1}$$

[1] There is no method for number theory.

[2] This extended some earlier work of Mordell, dealing with the case $g = 1$.

where $c(g,d)$ denotes an explicit positive constant depending only of the genus g of the curve and the degree d of the base field, and $\operatorname{rk}\operatorname{Jac}(C)(K)$ the rank of the finitely generated abelian group $\operatorname{Jac}(C)(K)$.

This is a major contribution to the central problem of number theory consisting in understanding the finiteness properties of solutions in rational numbers of systems of polynomial equations. In a few lines it is difficult to say anything precise concerning the proof of such a uniform bound, beyond that it involves a large amount of techniques from algebraic and Diophantine geometry and some very clever new ideas.

However, to put in perspective this result which provides an explicit bound on the cardinality of the finite set $C(K)$, I may emphasize that the existence of some effective control on the points of $C(K)$ themselves (for instance on their heights, or on the vacuity of $C(K)$) appears today completely out of reach. Actually, one cannot exclude that such an effective control does not exist in general, and from this perspective, the uniform bound (1) appears as a definitive result.

A second striking result of Vesselin Dimitrov is his positive solution of the *Schinzel-Zassenhaus conjecture.*

Consider a monic irreducible polynomial P in $\mathbb{Z}[X]$, of degree $n \geq 2$. A classical result of Kronecker asserts that, either P is a cyclotomic polynomial (or equivalently all its roots are roots of unity), or there exists a complex root α of P such $|\alpha| > 1$. The Schinzel-Zassenhaus conjecture asserts that the following quantitative form of this dichotomy holds: *for some positive universal constant* $c > 0$, *if* P *is not cyclotomic, then*

$$\max_{\alpha \in \mathbb{C}, P(\alpha)=0} \log |\alpha| \geq c/n. \tag{2}$$

Vesselin Dimitrov establishes this inequality with the optimal constant $c = (\log 2)/4$, answering positively a question open for more than half a century, and considered by the experts as completely out of reach in spite of (or because ?) of its elementary character.

The estimate (2) constitutes a breakthrough in the theory of *heights* — the real numbers attached to rational points of algebraic varieties over number fields that provide a measure of their arithmetic complexity. Since the work of Fekete, Szegö and Polya, properties of heights of rational points of the affine line are known to be closely related with potential theory in the plane.

Vesselin Dimitrov establishes the estimate (2) by an original application of a classical arithmetic rationality criterion on formal series with integral coefficients, involving the capacity of its domain of analyticity. A key point of the proof are precise capacity estimates concerning very special figures in the plane. His derivation of (2) is strikingly clever and elegant, and reveals that suitable "capacitary arguments" provide some new lines of attack to Diophantine problems considered till now as out of reach.

The recent joint work of Vesselin Dimitrov with Frank Calegari and Yunqing Tang, devoted to the proof of the *unbounded denominators conjecture*, provides another wonderful illustration of the relevance of such capacitary arguments to deep problems in arithmetic geometry.

The unbounded denominators conjecture asserts that, if a modular form f associated to some finite index subgroup Γ of $SL_2(\mathbb{Z})$ admits a q-expansion at the cusp $i\infty$ with integral coefficients, then f is actually a modular form associated to a congruence subgroup of $SL_2(\mathbb{Z})$:

$$\Gamma(N) := \big\{\gamma \in SL_2(\mathbb{Z}) \mid \gamma \equiv \mathrm{I}_2 \mod N\big\},$$

for some positive integer N.

The elementary formulation of this conjecture does not shed much light on its significance in arithmetic geometry. Let me simply recall that it has been formulated by Birch in the 70's, at the beginning of the revival of the theory of modular forms which eventually led to Wiles' proof of Fermat's conjecture, and stayed open till the breakthrough by Calegari, Dimitrov and Tang.

Let me also emphasize that the statement of this conjecture is in a sense extremely surprising: it relates two properties of "arithmeticity" concerning modular forms — admitting an integral q-expansion, and being associated to some congruence subgroup of $SL_2(\mathbb{Z})$ — which are related by some highly transcendental construction.

The proof by Calegari, Dimitrov and Tang intertwines very diverse threads. Subtle arithmetic and group theoretic results related to the congruence subgroup problem play a key role, and also delicate estimates involving potential theory in the plane and uniformization. All in all, their proof combines arguments originating from very different areas of arithmetic geometry in some highly unexpected and clever manner, and is awe inspiring by the fortitude required by their implementation into details.

For his impressive body of work, Vesselin Dimitrov has been awarded with the 2022 Oberwolfach Prize by the Oberwolfach Foundation.

Institut de Mathématique d'Orsay, Université Paris-Saclay, 91405 Orsay, France
jean-benoit.bost@math.u-psud.fr