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Finite and Infinite Dimensional Complex Geometry and Representation Theory

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Introduction by the Organisers

As the theme of the conference indicates, one of the organizers' main goals was to put together a group of participants with a wide range of interests in and around the complex geometric side of the representation theory of Lie groups and algebras. It is their belief that a hybrid approach to representation theory, in particular interaction between complex geometers and harmonic analysts standing on a strong foundation of finite- and infinite-dimensional Lie theory, will open up new avenues of thought and lead to progress in a number of areas.

Since the previous Oberwolfach conference (Represention Theory and Complex Analysis, April 2000), there has been quite a positive development toward these goals. A number of breakthroughs were achieved, and of course these were reported at this year's conference. More than half of the 47 participants (from 15 countries) are now working in some middle ground between complex geometry and representation theory. Furthermore, it was clear from the discussions both after the talks and in the evenings that most participants now understand each other's language well enough to discuss high level research projects.

A basic new component, infinite-dimensional complex geometry and related representation theory, was added this year. This quickly developing subject is already attracting wide attention. A goal for the future is to better integrate this with the more classical finite-dimensional theory.

One consequence of the broad range of backgrounds of the participants is that, without prodding by the organizers, virtually all speakers gave quite comprehensive introductions to their subjects before going into their most recent results. This was greatly appreciated by all!

Instead of attempting to summarize these talks we will let the following abstracts speak for themselves.

Workshop on Finite and Infinite Dimensional Complex Geometry and Representation Theory

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Abstracts

Visible actions on complex manifolds and multiplicity-one theorems

Toshiyuki Kobayashi

Multiplicity-free representations appear in various contexts such as Fourier transforms, Taylor series expansions, the Peter-Weyl theorem, branching laws for $GL_n \downarrow GL_{n+1}$, Clebsch-Gordan formula, Pieri's law, $GL_m - GL_n$ duality, the Plancherel formula for Riemannian symmetric spaces G/K, etc.

The aim of this talk is to report a simple principle based on complex geometry that explains the multiplicity-free property of various representations as above and more.

Suppose $\mathcal{V} \to D$ is an H-equivariant holomorphic vector bundle. Then, a representation of the group H is naturally defined on the Fréchet space $\mathcal{O}(D, \mathcal{V})$ of holomorphic sections. One asks:

"When does $\mathcal{O}(D, \mathcal{V})$ become multiplicity-free?"

We present a sufficient condition which comprises of a 'balance' of the base space D and fibers $\mathcal{V}_x$. To be more precise, let $P \to D$ be an H-equivariant principle K-bundle, $\mu : K \to GL_{\mathbb{C}}(V)$ a finite dimensional unitary representation, and $\mathcal{V} \simeq P \times_K V$. Suppose we are given automorphisms of Lie groups H and K, and a diffeomorphism of P, for which we use the same letter σ, satisfying the following two conditions:

$$\sigma(hpk) = \sigma(h)\sigma(p)\sigma(k)(h \in H; p \in P; k \in K),$$

The induced action of σ on $D(\simeq P/K)$ is anti-holomorphic.

For a subset B in P^σ, we define the following σ-stable subgroup

$$M := \{k \in K : bk \in Hb for any b \in B\}.$$

Theorem. *Assume that there exist σ and a subset B of P^σ satisfying the following three conditions:*

a) *HBK contains an interior point of P.*
b) *The restriction $\mu|_M$ decomposes as a multiplicity-free sum of irreducible representations of M.*

We shall write the decomposition as $\mu|_M \simeq \bigoplus_i \nu^{(i)}$.

c1) *$\mu \circ \sigma$ is isomorphic to μ^* (the contragredient representation of μ) as representations of K.*
c2) *$\nu^{(i)} \circ \sigma$ is isomorphic to $(\nu^{(i)})^*$ as representations of M for every i.*

Then, for any (abstract) unitary representation π of H which can be realized as a subrepresentation of $\mathcal{O}(D, \mathcal{V})$, π is multiplicity-free as an H-module.

Loosely speaking, our theorem asserts that the multiplicity-free property propagates from the smaller group M acting on fibers (see (b)) to the larger group H acting on holomorphic sections under a suitable condition (see (a)) on the H-action on the complex manifold D.

In light of the geometric condition (a) given in Theorem, we introduce the following notion:

Definition. The action of a Lie group H on a connected complex manifold D is **visible** if there exists a totally real submanifold N which meets generic H-orbit on D and satisfies

$$J(T_xN) \subset T_x(H \cdot x) \quad \text{for all } x \in N.$$

Example. 1) The natural action of $\mathbb{T}^n$ on the projective space $\mathbb{P}^{n-1}\mathbb{C}$ is visible.

2) The natural action of the direct product group $U(n_1) \times U(n_2) \times U(n_3)$ on the Grassmann variety $Gr_p(\mathbb{C}^n)(n = n_1 + n_2 + n_3 = p + q)$ is visible if $\min(n_1 + 1, n_2 + 1, n_3 + 1, p, q) \leq 2$.

3) Let G be a compact Lie group, and $G_{\mathbb{C}}$ its complexification. Then the action of $G \times G$ on $G_{\mathbb{C}}$ is visible.

4) Let $\mathcal{N}$ be a nilpotent orbit of $GL(n, \mathbb{C})$ corresponding to a partition $2^p 1^{n-2p}$. Then the action of $U(n)$ on $\mathcal{N}$ is visible for any p.

5) Let G/K be a Riemannian symmetric space of the non-compact type, and Ω its crown in $G_{\mathbb{C}}/K_{\mathbb{C}}$. Then the action of G on Ω is visible.

The above examples lead us to various kinds of multiplicity free representations. For example, (1) gives rise to the multiplicity-free property of the restriction $GL_n \downarrow GL_{n-1}$ as well as the Pieri rule for tensor product representations; (2) does to the list of all multiplicity-free tensor product representations of GL_n, which Stembridge found by a completely different method based on combinatorial argument; (3) does to the multiplicity-free property of the Peter-Weyl theorem of $L^2(G)$; (4) does to spherical nilpotent orbits whose complete list was recently given by Panyushev.

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A new look at the Maslov index

Bent Ørsted

The Maslov index is an invariant that appears several places in mathematics; roughly speaking it encodes qualitive aspects of solutions to certain variational problems - this includes asymptotic solutions to partial differential equations and flows of Hamiltonian systems. It also appears in the study of Lagrangian subspaces of a fixed symplectic vector space, where it gives an integer invariant for each triple of such subspaces. In this lecture we give several new ways of looking at the Maslov index, generalizing to the setting of bounded symmetric domains and defining a Maslov index for transversal triples of points in the Shilov boundary. This is done by integrating the canonical Kähler form over geodesic triangles in the domain and taking a limit to the boundary. We also extend to the infinite-dimensional situation and define a Maslov map from transversal triples on an appropriate Shilov boundary to the first homotopy group of the stabilizer of a base point in the domain. A crucial identity is shown in the context of Jordan triple systems, which gives a good algebraic framework for the infinite-dimensional case of such generalized flag manifolds and their invariants. This represents joint work, partly in progress, with J.-L. Clerc, K-H. Neeb, and W. Bertram.

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Analysis on the crown of a Riemannian symmetric space
Jacques Faraut

The crown of a Riemannian symmetric space $\mathcal{X} = G/K$ of non-compact type is a domain $\mathcal{D}$ in its complexification $\mathcal{X}_{\mathbb{C}} = G_{\mathbb{C}}/K_{\mathbb{C}}$, which has been intoduced by Akhiezer and Gindikin [1990]. It is also called the Akhiezer-Gindikin domain. It is interesting from various points of view: Riemannian geometry, complex geometry, analysis. From the analytic viewpoint it has the following remarkable property: All eigenfunctions of the invariant differential operators have a holomorphic extension to the crown $\mathcal{D}$, and the domain $\mathcal{D}$ is maximal for this property.

Consider the Cartan decomposition of $\mathfrak{g} = Lie(G)$, $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, and let $\mathfrak{a} \subset \mathfrak{p}$ be a Cartan subspace. Define

$$\omega = \{H \in \mathfrak{a} \mid \forall \alpha \in \Delta(\mathfrak{g}, \mathfrak{a}),\ |\alpha(H)| < \frac{\pi}{2}\}.$$

The crown can be described as

$$\mathcal{D} = G \exp i\omega \cdot o \quad (o = eK_{\mathbb{C}}).$$

On the other hand consider an Iwasawa decomposition $\mathcal{X} = NA \cdot o$, and define

$$\Xi = \bigcap_{k \in K} kN_{\mathbb{C}}A_{\mathbb{C}} \cdot o.$$

Theorem
The crown $\mathcal{D}$ is equal to the connected component Ξ_0 of Ξ which contains $\mathcal{X}$.

The inclusion $\mathcal{D} \subset \Xi_0$ has been proved by Krötz and Stanton for classical groups G [2001], and by Huckleberry in general [2002]. The reverse inclusion $\Xi_0 \subset \mathcal{D}$ has been proved by Barchini [2003].

If the symmetric space $\mathcal{X}$ is Hermitian, then $\mathcal{D} = \mathcal{X} \times \bar{\mathcal{X}}$ ([Huckleberry,2002], [Burns-Halverscheid-Hind,2003]). Let $\mathrm{Aut}(\mathcal{D})$ be the group of all holomorphic automorphisms of the crown of $\mathcal{D}$. In all cases $G \subset \mathrm{Aut}(\mathcal{D})$. In case of equality one says that $\mathcal{D}$ is rigid. Then $\mathcal{D}$ is either rigid or Hermitian ([Burns-Halverscheid-Hind,2003]).

Corollary

Every eigenfunction of all invariant differential operators has a holomorphic extension to the crown $\mathcal{D}$, and $\mathcal{D}$ is maximal for this property.

Such a joint eigenfunction f has a Poisson integral representation over the maximal boundary B of $\mathcal{X}$:

$$f(x) = \int_B P_\lambda(x,b)dT(b) \quad (\lambda \in \mathfrak{a}^*_{\mathbb{C}}),$$

where T is an analytic functional on $B = K/M$ (M is the centralizer of A in K). The Poisson kernel $P_\lambda(x,b)$ is related to the Iwasawa decomposition as follows. If $x = n \exp H \cdot o$ ($n \in N$, $H \in \mathfrak{a}$) one writes $H = \mathcal{A}(x)$. Then

$$P_\lambda(x,b) = e^{\langle \rho-\lambda, \mathcal{A}(k^{-1}x)\rangle} \quad (b = kM).$$

By [Clerc,1988],

$$e^{\langle \lambda, \mathcal{A}(x)\rangle} = \prod_{j=1}^{\ell} \psi_j(x)^{\lambda_j},$$

where ψ_j is a holomorphic function on $\mathcal{X}_{\mathbb{C}}$ which does not vanish on $N_{\mathbb{C}}A_{\mathbb{C}} \cdot o$. Since the crown $\mathcal{D}$ is simply connected, if follows that the function $x \mapsto P_\lambda(x,b)$ has a holomorphic extension to $\mathcal{D}$.

On the other hand, for any point z_0 on the boundary of the crown $\mathcal{D}$, one can find $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ and $b \in B$ such that the function $z \mapsto P_\lambda(z,b)$ has a singularity at z_0.

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Hardy spaces for the most continuous spectrum
Bernhard Krötz

We report on joint work with Simon Gindikin and Gestur Ólafsson (cf. [GKÓ02]).

Holomorphic extensions and boundary value maps have been valuable tools to solve problems in representation theory and harmonic analysis on *real* symmetric spaces. Two of the best known constructions are Hardy spaces with their boundary value maps and Cauchy-Szegö-kernels, and Fock space constructions with their corresponding Segal-Barmann transform. It is in this flavour that we establish a correspondence between eigenfunctions on a Riemannian symmetric spaces $X = G/K$ and a non-compactly causal (NCC) symmetric spaces $Y = G/H$ in this talk. In particular we, via analytic continuation, relate a *spherical function* ϕ_λ on G/K to a *holomorphic* H-invariant distribution on G/H.

Let us explain our results in more detail. On the geometric level we construct a certain minimal G-invariant Stein domain $\Xi_H \subseteq X_{\mathbb{C}} = G_{\mathbb{C}}/K_{\mathbb{C}}$ with the following properties: The Riemannian symmetric space X is embedded into Ξ_H as a totally real submanifold and the affine non-compactly causal space Y is isomorphic to the distinguished (Shilov) boundary of Ξ_H.

The minimal tube Ξ_H is a subdomain of the complex crown $\Xi \subseteq X_{\mathbb{C}}$ of X – an object first introduced in [AG90] which became subject of intense study over the last few years. A consequence is that all $\mathbb{D}(X)$-eigenfunctions on X extend holomorphically to Ξ_H [KS01b]. Another key fact is that $\mathbb{D}(X) \simeq \mathbb{D}(Y)$. Thus by taking limits on the boundary Y we obtain a realization of the $\mathbb{D}(X)$-eigenfunctions on X as $\mathbb{D}(Y)$-eigenfunctions on Y. Conversely, eigenfunctions on Y which holomorphically extend to Ξ_H yield by restriction eigenfunctions on X.

It seems to us that the above mentioned transition between eigenfunctions on X and Y is most efficiently described using the techniques from representation theory. To fix the notation let $(\pi, \mathcal{H})$ denote an admissible Hilbert representation of G with finite length. We write $\mathcal{H}^K$ for the space of K-fixed vectors and $(\mathcal{H}^{-\infty})^H$ for the space of H-fixed distribution vectors of π. Using the method of analytic continuation of representations as developed in [KS01a] we establish a bijection

$$\mathcal{H}^K \to (\mathcal{H}^{-\infty})^H_{\mathrm{hol}}, \quad v_K \mapsto v_H$$

where $(\mathcal{H}^{-\infty})^H_{\mathrm{hol}} \subseteq (\mathcal{H}^{-\infty})^H$ denotes the subspace characterized through the property that associated matrix coefficients on Y extend holomorphically to Ξ_H.

We give an application of our theory towards the geometric realization of the most-continuous spectrum $L^2(Y)_{\mathrm{mc}}$ of $L^2(Y)$. First progress in this direction was achieved in [GKÓ01]. There, for the cases where $\Xi = \Xi_H$, we defined a Hardy space $\mathcal{H}^2(\Xi)$ on Ξ and showed that there is an isometric boundary value mapping realizing $\mathcal{H}^2(\Xi)$ as a multiplicity one subspace of $L^2(Y)_{\mathrm{mc}}$ of full spectrum. It was an open problem how to define Hardy spaces for general NCC symmetric spaces

Y and to determine the Plancherel measure explicitely. We solve this problem by giving a spectral definition of the Hardy space, i.e., we take the conjectured Plancherel measure and define a Hilbert space of holomorphic functions $\mathcal{H}^2(\Xi_H)$ on Ξ_H. The identification of $\mathcal{H}^2(\Xi_H)$ as a Hardy space then follows by establishing an isometric boundary value mapping $b : \mathcal{H}^2(\Xi_H) \hookrightarrow L^2(G/H)_{\mathrm{mc}}$. In particular we achieve a geometric realization of a multiplicity free subspace of $L^2(Y)_{\mathrm{mc}}$ in holomorphic functions.

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Structure of $gl(\infty)$

Ivan Dimitrov

Let U and V be two (infinite dimensional) complex vector spaces with a non–degenerate pairing $\langle \circ, \circ \rangle : U \times V \to \mathbb{C}$. Consider the Lie algebra $\mathfrak{g} := U \otimes V$. When both U and V are countable dimensional, $\mathfrak{g}$ is isomorphic to the Lie algebra $gl(\infty)$ of finitary infinite matrices, see [M]. A maximal locally solvable subalgebra of $\mathfrak{g}$ is called a *Borel subalgebra of* $\mathfrak{g}$. In this talk we describe the Borel subalgebras of $\mathfrak{g}$ and discuss their relation with maximal toral subalgebras of $\mathfrak{g}$.

In order to describe the Borel subalgebras of $\mathfrak{g}$ we need the notion of a generalized flag in U introduced in [DP]. A chain $\mathcal{F} = \{F'_\alpha, F''_\alpha\}_{\alpha \in A}$ of subspaces of U is a *generalized flag in* U if F'_α is the immediate predecessor of F''_α and $U \backslash \{0\} = \cup_\alpha F''_\alpha \backslash F'_\alpha$. (Here we allow $F'_\alpha = F''_\beta$.) For any chain $\mathcal{C}$ of subspaces of U, there is a canonical generalized flag $\mathit{fl}(\mathcal{C})$ associated with $\mathcal{C}$. The pairing between U and V defines the closure operation on subspaces of U given by $\bar{H} := H^{\perp\perp}$. This operation was first introduced and studied by Mackey in his thesis, see also [M]. For any generalized flag $\mathcal{F}$ in U we define the closure $\bar{\mathcal{F}}$ of $\mathcal{F}$ as $\mathit{fl}(\mathcal{F}^{\perp\perp})$, where $\mathcal{F}^{\perp\perp}$ denotes the chain in U consisting of the closures of all subspaces in $\mathcal{F}$. $\mathcal{F}$ is a *closed generalized flag in* U if $\bar{\mathcal{F}} = \mathcal{F}$, and $\mathcal{F}$ is a *strongly closed generalized flag in* U if $\mathcal{F}^{\perp\perp} = \mathcal{F}$. Clearly, any strongly closed generalized flag in U is closed. $\mathcal{F}$ is closed if and only if $\bar{F}''_\alpha = F''_\alpha$ and $\bar{F}'_\alpha$ equals either F'_α or F''_α. For any generalized flag $\mathcal{F}$ in U the subalgebra of $\mathfrak{g}$ which stabilizes $\mathcal{F}$ is $\mathrm{St}_{\mathcal{F}} = \sum_\alpha F''_\alpha \otimes (F'_\alpha)^\perp$. The following theorem describes the Borel subalgebras of $\mathfrak{g}$.

Theorem 1. *The map $\mathcal{F} \mapsto \mathrm{St}_{\mathcal{F}}$ establishes a bijection between maximal closed generalized flags in U and Borel subalgebras of $\mathfrak{g}$.*

This theorem provides a rather explicit description of all Borel subalgebras of $\mathfrak{g}$. The results are most interesting in the case when both U and V are countable dimensional, i.e. $\mathfrak{g} \simeq gl(\infty)$. In this case we can represent $\mathfrak{g}$ as the direct limit $\varinjlim \mathfrak{g}_n$, where $\mathfrak{g}_n \simeq gl(n)$. It is clear that choosing a direct system of Borel subalgebras $\mathfrak{b}_n$ of $\mathfrak{g}_n$, the limit subalgebra $\mathfrak{b} := \varinjlim \mathfrak{b}_n$ is necessarily a Borel subalgebra of $\mathfrak{g}$. The converse, however, is not true. In fact we have the following theorem.

Theorem 2. *A Borel algebra $\mathfrak{b}$ of $\mathfrak{g}$ is the direct limit of Borel algebras $\mathfrak{b}_n$ of $\mathfrak{g}_n$ for some (but not every) direct system $\mathfrak{g} = \varinjlim \mathfrak{g}_n$, such that $\mathfrak{g}_n \simeq gl(n)$, if and only if the maximal closed generalized flags corresponding to $\mathfrak{b}$ both in U and in V are strongly closed.*

Finally, we consider the relationship between maximal toral subalgebras of $\mathfrak{g}$ and Borel subalgebras of $\mathfrak{g}$. We prove that, for any $\mathfrak{b} \subset \mathfrak{g}$, there exists a maximal toral subalgebra $\mathfrak{t} \subset \mathfrak{b}$ which is the compliment (as a vector space) of the locally nilpotent radial of $\mathfrak{b}$, i.e. $\mathfrak{b} = \mathfrak{t} \oplus \mathfrak{n}$, where $\mathfrak{n}$ is the locally nilpotent radical of $\mathfrak{b}$. Furthermore, we establish another criterion for $\mathfrak{b} = \varinjlim \mathfrak{b}_n$ as in Theorem 2. To state it we need to recall the definition of a splitting maximal toral subalgebra of $\mathfrak{g}$. A maximal toral subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ is called *splitting* if it acts locally finitely on $\mathfrak{g}$, equivalently, if $\mathfrak{g}$ admits a root decomposition with respect to $\mathfrak{t}$. (For more details on maximal toral subalgebras of $\mathfrak{g}$ see [NP].) We then prove that the conditions of Theorem 2 are equivalent to the requirement that $\mathfrak{b}$ contains a splitting maximal toral subalgebra of $\mathfrak{g}$.

The talk is based on a joint work with Ivan Penkov.

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Dolbeault cohomology of a loop space

László Lempert
(joint work with Ning Zhang (Riverside))

Loop spaces LM of compact complex manifolds M promise to have rich analytic cohomology theories, and it is expected that sheaf and Dolbeault cohomology groups of LM will shed new light on the complex geometry and analysis of M itself. This idea first occurs in [W], in the context of the infinite dimensional Dirac operator, and then in [HBJ] that touches upon Dolbeault groups of loop spaces; but in all this both works stay heuristic. Our goal here is to present rigorous results concerning the $H^{0,1}$ Dolbeault group of the first interesting loop space, that of the Riemann sphere $\mathbb{P}_1$. One noteworthy fact that emerges from this research is that analytic cohomology of loop spaces, unlike topological cohomology, is very sensitive to the regularity of loops admitted in the space. Another concerns local functionals, a notion from theoretical physics. Roughly, if M is a manifold, a local functional on a space of loops $x\colon S^1 \to M$ is one of form

$$f(x) = \int_{S^1} \Phi(t, x(t), \dot{x}(t), \ddot{x}(t), \dots)dt,$$

where Φ is a function on $S^1\times$ an appropriate jet bundle of M. It turns out that all cohomology classes in $H^{0,1}(L\mathbb{P}_1)$ are given by local functionals. Nonlocal cohomology classes exist only perturbatively, i.e., in a neighborhood of constant loops in $L\mathbb{P}_1$; but none of them extends to the whole of $L\mathbb{P}_1$.

We fix a smoothness class C^k, $k = 1, 2, \dots, \infty$, or Sobolev $W^{k,p}$, $k = 1, 2, \dots, 1 \le p < \infty$. If M is a finite dimensional complex manifold, consider the space $LM = L_kM$ resp. $L_{k,p}M$ of maps $S^1 = \mathbb{R}/\mathbb{Z} \to M$ of the given regularity. These spaces are complex manifolds modeled on a Banach space, except for $L_\infty M$, which is modeled on a Fréchet space. We shall focus on the loop space(s) $L\mathbb{P}_1$. As on any complex manifold, one can consider the space $C^\infty_{r,q}(L\mathbb{P}_1)$ of smooth (r, q) forms, the operators $\bar\partial : C^\infty_{r,q}(L\mathbb{P}_1) \to C^\infty_{r,q+1}(L\mathbb{P}_1)$, and the associated Dolbeault groups $H^{r,q}(L\mathbb{P}_1)$; for all this, see e.g. [L1,2]. On the other hand, let $\mathfrak{F}$ be the space of holomorphic functions $F\colon \mathbb{C} \times L\mathbb{C} \to \mathbb{C}$ that have the following properties:

(1) $F(\zeta/\lambda, \lambda^2 y) = O(\lambda^2)$, as $\mathbb{C} \ni \lambda \to 0$;
(2) $F(\zeta, x + y) = F(\zeta, x) + F(\zeta, y)$, if $\operatorname{supp} x \cap \operatorname{supp} y = \emptyset$;
(3) $F(\zeta, y + \text{const}) = F(\zeta, y)$.

As we shall see, the additivity property (2) implies $F(\zeta, y)$ is local in y.

Theorem 1. $H^{0,1}(L\mathbb{P}_1) \approx \mathbb{C} \oplus \mathfrak{F}$.

In the case of $L_\infty\mathbb{P}_1$, examples of $F \in \mathfrak{F}$ are

$$F(\zeta, y) = \zeta^\nu \Big\langle \Phi, \prod_{j=0}^{m} y^{(d_j)} \Big\rangle, \tag{1}$$

where Φ is a distribution on $S^1, y^{(d)}$ denotes d'th derivative, each $d_j \geq d_0 = 1$, and $0 \leq \nu \leq 2m$. A general function in $\mathfrak{F}$ can be approximated by linear combinations of functions of form (1).

This brings us to the issue of topology on $H^{0,1}(L\mathbb{P}_1)$ and on $\mathfrak{F}$. On any, possibly infinite dimensional complex manifold X the space $C^\infty_{r,q}(X)$ can be given the compact–C^∞ topology as follows. First, the compact–open topology on $C^\infty_{0,0}(X) = C^\infty(X)$ is generated by C^0 seminorms $\|f\|_K = \sup_K |f|$ for all $K \subset X$ compact. The family of C^ν seminorms is defined inductively: each $C^{\nu-1}$ seminorm $\| \quad \|$ on $C^\infty(TX)$ induces a C^ν seminorm $\|f\|' = \|df\|$ on $C^\infty(X)$. The collection of all C^ν seminorms, $\nu = 0, 1, \ldots$, defines the compact–C^∞ topology on $C^\infty(X)$. The compact–C^∞ topology on a general $C^\infty_{r,q}(X)$ is induced by the embedding $C^\infty_{r,q}(X) \subset C^\infty(\overset{r+q}{\oplus} TX)$. With this topology $C^\infty_{r,q}(X)$ is a separated locally convex vector space, complete if X is first countable. The quotient space $H^{r,q}(X)$ inherits a locally convex topology, not necessarily separated. We note that on the subspace $\mathcal{O}(X) \subset C^\infty(X)$ of holomorphic functions the compact–C^∞ topology restricts to the compact–open topology. The isomorphism in Theorem 1 is topological; it is also equivariant with respect to the obvious actions of the group of C^k diffeomorphisms of S^1.

There is another group, the group $G \approx \mathrm{PSL}(2, \mathbb{C})$ of holomorphic automorphisms of $\mathbb{P}_1$, whose holomorphic action on $L\mathbb{P}_1$ (by post–composition) and on $H^{0,1}(L\mathbb{P}_1)$ will be of greater concern to us. Theorems 2, 3, 4 below will describe the structure of $H^{0,1}(L\mathbb{P}_1)$ as a G–module. Recall that any irreducible (always holomorphic) G–module is isomorphic, for some $n = 0, 1, \ldots$, to the space $\mathfrak{K}_n$ of holomorphic differentials $\psi(\zeta)(d\zeta)^{-n}$ of order $-n$ on $\mathbb{P}_1$; here ψ is a polynomial, deg $\psi \leq 2n$, and G acts by pullback. The n'th isotypical subspace of a G–module V is the sum of all irreducible submodules isomorphic to $\mathfrak{K}_n$. In particular, the 0'th isotypical subspace is the space V^G of fixed vectors.

Theorem 2. *If $n \geq 1$, the n'th isotypical subspace of $H^{0,1}(L_\infty\mathbb{P}_1)$ is isomorphic to the space $\mathfrak{F}^n$ spanned by functions of form (0.1), with $m = n$.*

The fixed subspace of $H^{0,1}(L\mathbb{P}_1)$ can be described more explicitly, for any loop space:

Theorem 3. *The space $H^{0,1}(L\mathbb{P}_1)^G$ is isomorphic to $C^{k-1}(S^1)^*$ resp. $W^{k-1,p}(S^1)^*$, if the dual spaces are endowed with the compact–open topology.*

The isomorphisms in Theorem 3 are not Diff S^1 equivariant. To remedy this, one is led to introduce the spaces $C^l_r(S^1)$ resp. $W^{l,p}_r(S^1)$ of differentials $y(t)(dt)^r$ of order r on S^1, of the corresponding regularity; $L^p_r = W^{0,p}_r$. Then $H^{0,1}(L\mathbb{P}_1)^G$ will be Diff S^1 equivariantly isomorphic to $C^{k-1}_1(S^1)^*$, resp. $W^{k-1,p}_1(S^1)^*$.

For low regularity loop spaces one can very concretely represent all of $H^{0,1}(L\mathbb{P}_1)$:

Theorem 4. *(a) If $1 \leq p < 2$, all of $H^{0,1}(L_{1,p}\mathbb{P}_1)$ is fixed by G, hence it is isomorphic to $L^{p'}(S^1)$, with $p' = p/(p-1)$.*

(b) If $1 \le p < \infty$ then $H^{0,1}(L_{1,p}\mathbb{P}_1)$ is isomorphic to

$$\bigoplus_{0\le n\le p-1} \mathfrak{K}_n \otimes L^{p/(n+1)}_{n+1}(S^1)^* \approx \bigoplus_{0\le n\le p-1} \mathfrak{K}_n \otimes L^{p_n}_{-n}(S^1), \qquad p_n = \frac{p}{p-1-n},$$

and so it is the sum of its first $[p]$ isotypical subspaces. Indeed, the isomorphisms above are $G \times$ Diff S^1 equivariant, G, resp. Diff S^1 acting on one of the factors $\mathfrak{K}_n$, L^q_r naturally, and trivially on the other.

Again, the dual spaces are endowed with the compact–open topology.

To finish this write up, here is a list of relevant literature:

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Infinite-Dimensional Homogeneous Spaces and Operator Ideals

Daniel Beltiţă

The existence of invariant Kähler structures on homogeneous spaces of certain Lie groups turns out to be a phenomenon that is not confined to finite dimensions. Our research concerns this phenomenon in the case of some classes of infinite-dimensional Lie groups associated with ideals of compact operators on Hilbert spaces.

More specifically, we have introduced in the paper [Be03] a notion of admissible pair of operator ideals $(\mathfrak{I}_0, \mathfrak{I}_1)$ and have used it to construct Kähler homogeneous spaces of Banach-Lie groups naturally associated with such pairs. One special instance of admissible pair is a pair of Schatten ideals $(\mathfrak{S}_p, \mathfrak{S}_q)$, where $2 \le p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. More generally, certain dual pairs of Lorentz ideals also turn out to be admissible.

Here is the precise definition of an admissible pair:

Definition. Let $\mathcal{H}$ be a complex Hilbert space and $\mathcal{B}(\mathcal{H})$ the Banach algebra of all bounded linear operators on $\mathcal{H}$. An *admissible pair* of ideals of $\mathcal{B}(\mathcal{H})$ is a pair $(\mathfrak{I}_0, \mathfrak{I}_1)$ of two-sided ideals of $\mathcal{B}(\mathcal{H})$ satisfying the following conditions:

(a) The ideal $\mathfrak{I}_0$ is equipped with a norm $\|\cdot\|_{\mathfrak{I}_0}$ making it into a reflexive separable Banach space satisfying

$$\|T\| \le \|T\|_{\mathfrak{I}_0} = \|T^*\|_{\mathfrak{I}_0} \text{ and } \|ATB\|_{\mathfrak{I}_0} \le \|A\| \cdot \|T\|_{\mathfrak{I}_0} \cdot \|B\|$$

whenever $A, B \in \mathcal{B}(\mathcal{H})$ and $T \in \mathfrak{I}_0$.

(b) We have $\mathfrak{I}_1 \cdot \mathfrak{I}_0 \subseteq \mathfrak{S}_1(\mathcal{H})$ and the bilinear functional

$$\mathfrak{I}_1 \times \mathfrak{I}_0 \to \mathbb{C}, \quad (K, T) \mapsto \mathrm{Trace}\,(KT)$$

induces a vector space isomorphism of $\mathfrak{I}_1$ onto the topological dual of the Banach space $(\mathfrak{I}_0, \|\cdot\|_{\mathfrak{I}_0})$, where $\mathfrak{S}_1(\mathcal{H})$ denotes the trace class on $\mathcal{H}$.

(c) We have $\mathfrak{I}_1 \subseteq \mathfrak{I}_0$.

Using the notion of admissible pair, one can construct infinite-dimensional Kähler manifolds as described in the following theorem. In this statement, for any operator ideal $\mathfrak{I}$ we denote by $\mathfrak{u}_{\mathfrak{I}} = \{T \in \mathfrak{I} \mid T^* = -T\}$ the Lie algebra of skew-adjoint operators in $\mathfrak{I}$, and we also denote by $\mathrm{U}_{\mathfrak{I}} = \{T \in \mathrm{id}_{\mathcal{H}} + \mathfrak{I} \mid T^*T = TT^* = \mathrm{id}_{\mathcal{H}}\}$ the group of all unitary operators in $\mathrm{id}_{\mathcal{H}} + \mathfrak{I}$.

Theorem. *Let $(\mathfrak{I}_0, \mathfrak{I}_1)$ be an admissible pair of ideals of $\mathcal{B}(\mathcal{H})$ and A a self-adjoint element of $\mathcal{B}(\mathcal{H})$. Consider the following objects:*

- $\mathrm{U}_{\mathfrak{I}_0,\mathfrak{I}_1}(A) = \{T \in \mathrm{U}_{\mathfrak{I}_0} \mid T^*AT \in A + \mathfrak{I}_1\} = \{T \in \mathrm{U}_{\mathfrak{I}_0} \mid [A, T] \in \mathfrak{I}_1\}$,
- $H_{\mathfrak{I}_0,A} = \{T \in \mathrm{U}_{\mathfrak{I}_0} \mid T^*AT = A\}$,
- $\mathfrak{u}_{\mathfrak{I}_0,\mathfrak{I}_1}(A) = \{T \in \mathfrak{u}_{\mathfrak{I}_0} \mid [A, T] \in \mathfrak{I}_1\}$,
- $\omega\colon \mathfrak{u}_{\mathfrak{I}_0,\mathfrak{I}_1}(A) \times \mathfrak{u}_{\mathfrak{I}_0,\mathfrak{I}_1}(A) \to \mathbb{R}, \quad \omega(T_1, T_2) = \mathrm{Trace}\,(\mathrm{i}[A, T_1]T_2)$.

Then the following assertions hold.

(a) *The group $\mathrm{U}_{\mathfrak{I}_0,\mathfrak{I}_1}(A)$ has a natural structure of connected real Banach-Lie group with the Lie algebra $\mathfrak{u}_{\mathfrak{I}_0,\mathfrak{I}_1}(A)$, and the bilinear functional ω is a continuous 2-cocycle of the real Banach-Lie algebra $\mathfrak{u}_{\mathfrak{I}_0,\mathfrak{I}_1}(A)$. Furthermore, $H_{\mathfrak{I}_0,A}$ is a Banach-Lie subgroup of $\mathrm{U}_{\mathfrak{I}_0,\mathfrak{I}_1}(A)$ whose Lie algebra equals $\{T \in \mathfrak{u}_{\mathfrak{I}_0,\mathfrak{I}_1}(A) \mid \omega(T, \cdot) \equiv 0\}$.*

(b) *The 2-cocycle ω induces a $\mathrm{U}_{\mathfrak{I}_0,\mathfrak{I}_1}(A)$-invariant weakly symplectic form Ω on the homogeneous space $\mathrm{U}_{\mathfrak{I}_0,\mathfrak{I}_1}(A)/H_{\mathfrak{I}_0,A}$.*

(c) *If the spectrum of the operator A is finite, then there exists a* $\mathrm{U}_{\mathfrak{J}_0,\mathfrak{J}_1}(A)$*-invariant complex structure making the weakly symplectic homogeneous space*
$(\mathrm{U}_{\mathfrak{J}_0,\mathfrak{J}_1}(A)/H_{\mathfrak{J}_0,A},\Omega)$ *into a weakly Kähler homogeneous space.*

We now outline the method used in [Be03] to construct the aforementioned Kähler structures. The main point is that we actually study Banach-Lie groups associated with admissible pairs and with certain n-tuples of self-adjoint operators. We use the joint functional calculus of those n-tuples (which is a special instance of the Weyl functional calculus) to construct Kähler polarizations in the complexified Lie algebras of the Lie groups under consideration. In fact, the polarizations arise as spectral subspaces corresponding to certain subsets of the joint spectrum of the corresponding n-tuple. A remarkable point of this approach is that it actually holds in a quite general setting. E.g., besides the homogeneous spaces of groups associated with operator ideals, that approach leads to complex structures on the flag manifolds associated with arbitrary associative unital Banach algebras.

We mention that certain special instances of the complex homogeneous spaces constructed by the above described method were already shown to play a significant role in representation theory of certain Hilbert-Lie groups associated with the Hilbert-Schmidt ideal (see e.g., [Bo80], [Ca85], [Ne00], [Ne02]). From this point of view, it is interesting to investigate the role played by the new classes of complex homogeneous spaces in the representation theory of more general Banach-Lie groups. On the other hand, it would be important to understand whether the specific properties of the operator ideals correspond to any particular phenomena in the complex geometry of the corresponding homogeneous spaces (compare also [Up85]).

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Realizing Lie Groups as Automorphism Groups of Complex Manifolds

Jörg Winkelmann

Let X be a hyperbolic (in the sense of Kobayashi) complex connected manifold. Then the group of all holomorphic automorphisms of X (endowed with the compact-open topology) is a finite-dimensional real Lie group with countably many connected components. This raises the question whether conversely every such Lie group can be realized as a the full automorphism group of a hyperbolic complex manifold.

We prove that this is true if the group is connected or discrete.

Theorem 1. *Let G be a (finite-dimensional) real connected Lie group or a countable discrete group.*

Then there exists a Stein hyperbolic connected complex manifold X such that G is isomorphic to the group of all automorphisms (i.e. biholomorphic selfmaps) of X.

The first step in this direction was the result for compact Lie groups. Saerens and Zame ([5]), and independently Bedford and Dadok ([1]) proved that, given a compact real Lie group K there always exists a strictly pseudoconvex bounded domain $D \subset \mathbb{C}^n$ such that $Aut(D) \simeq K$. By the theorem of Wong-Rosay (which states that every strictly pseudoconvex bounded domain with non-compact automorphism group is isomorphic to the ball) it is clear that an arbitrary non-compact real Lie group can not be realized as the automorphism of a strictly pseudoconvex bounded domain in $\mathbb{C}^n$. However, as proved in [8], for any connected real Lie group G there does exist a complex manifold X on which G acts effectively. Moreover, X can be chosen in such a way that it enjoys several of the key properties of strictly pseudoconvex bounded domains. Namely, X can be chosen such that it is both Stein and hyperbolic in the sense of Kobayashi.

In [10] we verified that one can rule out additional automorphisms, i.e. it is possible to achieve $Aut(X) \simeq G$. The precise result is the following:

Theorem 2. *Let G be a connected real Lie group. Then there exists a Stein, complete hyperbolic complex manifold X on which G acts effectively, freely, properly and with totally real orbits such that* $\mathrm{Aut}_{\mathcal{O}}(X) \simeq G$.

The idea is to follow the strategy of Saerens and Zame: Construct the desired manifold as an open subset of a larger Stein manifold in such a way that the given group acts on this open subset. Ensure that every automorphism of this open subset can be extended to the boundary, then modify the boundary in such a way that this CR-hypersurface simply has no automorphisms other than those from the given group. The latter can be done using the fact that a CR-hypersurface (unlike a complex manifold) does have local invariants. A principal difficulty in this approach is to obtain an extension of automorphisms of the open subset to the boundary. If one is concerned only with compact Lie groups, then one can work with a strictly pseudoconvex bounded domain D. For such a domain it is evident that for every automorphism ϕ of D there exists a sequence $x_n \in D$ such that both x_n and $\phi(x_n)$ converge to a strictly pseudoconvex point in the boundary. This is the starting point for the extension of the automorphism ϕ to the boundary ∂D.

Now, our goal is to obtain a result for arbitrary connected Lie groups, which are not necessarily compact.

This lack of compactness assumption creates some difficulties.

There are two main problems: First, an arbitrary non-compact Lie group is not necessarily linear. For instance, the universal cover of $SL_2(\mathbb{R})$ cannot be embedded into a linear group. Second, as already mentioned, the theorem of Wong-Rosay implies that in general a non-compact Lie group can not be realized as the full automorphism group of a strictly pseudoconvex bounded domain with smooth boundary. Thus we have to work with domains which are not bounded or where the boundary is not everywhere smooth. The trouble is that it is therefore no longer clear that for every automorphism ϕ there exists a sequence x_n in the domain such that both x_n and $\phi(x_n)$ converge to a nice point in the boundary.

In [7] a result similar to ours is claimed for certain Lie groups with a rather sketchy outline of a possible proof.

The first of the aforementioned two problems is dealt with by *assuming* the group G to be linear while the second problem is simply ignored. Since the second problem is in fact a serious obstacle, the proof sketched in [7] can not be regarded as complete.

We proceed in the following way: To deal with the first problem, we note that every Lie algebra is linear by the theorem of Ado. Therefore, in a certain sense, every Lie group is linear up to coverings and the first problem can be attacked by working carefully with coverings.

For the second problem, we use bounded domains whose boundaries are smooth outside an exceptional set E which is small in a certain sense. Exploiting this smallness we prove that for every automorphism ϕ there must exist a sequence x_n such that both x_n and $\phi(x_n)$ converge to a boundary point outside the "bad set" E.

Once this has been verified, we can prove (using arguments similar to those used in[5], [1]) that ϕ extends as holomorphic map near $\lim(x_n)$, and use the theory of Chern-Moser-invariants to deduce that ϕ was in fact given by left multiplication with an element of G.

For discrete groups the following statement is proved in [9]:

Theorem 3. *Let G be a countable discrete group. Then there exists a non-compact Riemann surface X, hyperbolic in the sense of Kobayashi, such that G is isomorphic to the automorphism group of X.*

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Principal Series Representations and Dirac Operators

Roger Zierau

Kostant defined a remarkable invariant differential operator in [7] which he called the cubic Dirac operator. Given a connected semisimple Lie group, a closed reductive subgroup and a homogeneous vector bundle $\mathcal{E} \to G/H$ of finite rank, the cubic Dirac operator is a G-invariant differential operator on sections:

$$D : C^\infty(G/H, \mathcal{E} \otimes \mathcal{S}) \to C^\infty(G/H, \mathcal{E} \otimes \mathcal{S}). \tag{1}$$

Here S is the spin representation of $\mathfrak{h}$. In this lecture we discuss joint work with Salah Mehdi in which the kernel of D is studied when H is noncompact and $\mathrm{rank}(\mathfrak{g}) = \mathrm{rank}(\mathfrak{h})$. The main result is an integral formula for certain solutions of $Df = 0$. In particular, the kernel is nonzero and certain interesting representations occur.

The cubic Dirac operator is defined as follows. There is an orthogonal decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ with respect to the Killing form of $\mathfrak{g}$ (however, we need to assume the Killing form on $\mathfrak{h}$ is nondegenerate). Then $\mathfrak{q}$ is equipped with a nondegenerate (possibly indefinite) symmetric form. Thus, one may build a corresponding Clifford algebra and spin representation of $\mathfrak{so}(\mathfrak{q})$. Since $ad : \mathfrak{h} \to \mathfrak{so}(\mathfrak{q})$ we obtain the representation $\sigma \circ \mathrm{ad}$ of $\mathfrak{h}$, which we call the spin representation of $\mathfrak{h}$. In (1) we require only that E is a representation of $\mathfrak{h}$ so that $E \otimes S$ integrates to a representation of H. Then $\mathcal{E} \otimes \mathcal{S} \to G/H$ is the corresponding homogeneous vector bundle. Now choose a basis $\{X_j\}$ of $\mathfrak{q}$ so that $\langle X_j\,, X_k\rangle_{\mathfrak{q}} = \epsilon_j \delta_{jk}$, with $\epsilon_j = \pm 1$. Let $c \in Cl(\mathfrak{q})$ correspond to the alternating 3-form $\langle X\,, [Y, Z]\rangle_{\mathfrak{q}}$ on $\mathfrak{q}$. The cubic Dirac operator of (1) is defined by

$$D = \sum_j \epsilon_j r(X_j) \otimes 1 \otimes \gamma(X_j) + 1 \otimes 1 \otimes \gamma(c). \tag{2}$$

Here γ denotes Clifford multiplication and $r(X)$ is the right action of $X \in \mathfrak{g}$ on functions.

There are several well-known cases where such an operator has been studied. Most notably, when H is a maximal compact subgroup of G, then $c = 0$ and D is the 'usual' Dirac operator arising from an invariant connection. In this case, the kernel of D (on L_2-sections) is a relative discrete series representation and all relative discrete series representations of G occur this way. See [11], [1] and [12]. Another case is when G is compact. Then, in [8] and [9] the kernel of D is seen to be an irreducible G-representation. This is a generalization of the Bott-Borel-Weil Theorem. A remarkable property of D which relates D to infinitesimal character is contained in [6].

Now let us turn to a noncompact group G and noncompact reductive subgroup H. Let E and S be as above. Our goal is to study the kernel of D and our approach is to find a G-intertwining map from a principal series representation of G into $Ker(D)$.

We briefly describe the construction. Let θ be a Cartan involution of $\mathfrak{g}$ which stabilizes $\mathfrak{h}$ and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{s}$ be the corresponding Cartan decomposition of $\mathfrak{g}$. The principal series consists of representations induced from representations of real parabolic subgroups of G. Our subgroup H determines a parabolic subgroup as follows. Choose a maximal abelian subalgebra $\mathfrak{a}$ in $\mathfrak{h} \cap \mathfrak{s}$. Then $\mathfrak{a}$ determines a parabolic $P = MAN$ (up to a choice of N). Note that it is important here that $\mathfrak{g}_{\mathbf{C}}$ and $\mathfrak{h}_{\mathbf{C}}$ have the same ranks. It follows that $P \cap H = (M \cap H)A(N \cap H)$ is a minimal parabolic subgroup of H. In particular $H \cap K \cdot eP = H \cdot eP$ is a closed H-orbit in G/P.

Lemma 3. *Each relative discrete series representation of M occurs in the kernel of*
$$D_{M/M\cap H} : \mathbf{C}^\infty(M/M \cap H, \mathcal{F} \otimes S) \to \mathbf{C}^\infty(M/M \cap H, \mathcal{F} \otimes S)$$
for some homogeneous bundle $\mathcal{F} \to M/M \cap H$. Note that, with our choice of P, $M \cap H$ is compact.

This Lemma is of course related to the results on the discrete series mentioned above. However, here we are not concerned with the L_2 statement; by relative discrete series here we mean a representation infinitesimally equivalent to a relative discrete series representation.

For a representation W of P we write $C^\infty(G/P, \mathcal{W})$ for the induced representation (the smooth principal series representation).

Lemma 4. *For any smooth representation W of P, given some nonzero $t \in Hom_{P\cap H}(W \otimes \mathbf{C}_{\rho+2\rho_{\mathfrak{h}}}, E \otimes S)$ there is a nonzero G-intertwining map*
$$P_t : C^\infty(G/P, \mathcal{W} \otimes \mathbf{C}_{\rho_{\mathfrak{h}}}) \to \mathbf{C}^\infty(G/H, \mathcal{E} \otimes S)$$
$$(P_t\phi)(g) = \int_{H\cap K} \ell \cdot (\phi(g\ell))\, d\ell.$$

Therefore, we need to find a W and t so that the image of P_t lies in the kernel of D. This is accomplished by finding a relative discrete series representation W of M so that, when realizing W as $Ker(D_{M/M\cap H})$ as in Lemma 3, t is evaluation at $e \in M$ and the following holds.

Theorem 5. *When the highest weight μ of E is sufficiently regular, the image of P_t lies in $Ker(D)$.*

Remark 6. Note the analogy between our construction and that of the Poisson integral. The Poisson integral is a formula giving harmonic functions on the unit disk in $\mathbf{C}$. In fact, the generalization of this is the Poisson transform (see, for example, [5, Ch. II, Section 4.1]) producing joint eigenfunctions of the G-invariant differential operators on the riemannian symmetric space G/K. One notes that the Poisson transform is an integral over the boundary G/P of G/K and the formula comes from an analogue of Lemma 4 with S and E replaced by the trivial representation. In our setting, $H \cap K \cdot eP = H \cdot eP \subset G/P$. We may therefore say that integration over 'a piece of the boundary' of G/H gives solutions to the Dirac equation $Df = 0$.

Remark 7. The results discussed here may be viewed as a generalization of [3], [2] and [4] in the following sense. If G/H is a measurable open orbit in a flag variety (i.e., an elliptic coadjoint orbit), then $D = \overline{\partial} + \overline{\partial}^*$. In this case, the operator initially studied in [3] coincides with the intertwining operator P_t above.

Remark 8. The principal series representations are fairly well understood. Thus, certain representatins occurring in $Ker(D)$ can be identified via the Langlands classification. Furthermore, the growth of harmonic spinors of the form $P_t\phi$ can be studied by considering properties of ϕ and using techniques of Harish-Chandra.

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Theta lifting of unitary lowest weight representations and their associated cycles

Kyo Nishiyama

We consider a reductive dual pair (G, G') in the stable range with G' the smaller member and of Hermitian symmetric type. Namely, the following three kinds of dual pairs will be treated.

	the pair (G, G')	stable range condition
Case $\mathbb{R}$:	$(O(p,q), Sp(2n,\mathbb{R}))$	$2n < \min(p,q)$
Case $\mathbb{C}$:	$(U(p,q), U(m,n))$	$m+n \leq \min(p,q)$
Case $\mathbb{H}$:	$(Sp(p,q), O^*(2n))$	$n \leq \min(p,q)$

We study the theta lifting of a unitary lowest weight representation π' of G', which may be singular. The main result is an explicit determination of the associated cycle of the lifted representation $\theta(\pi')$. More precisely, we prove that

$$\theta(\mathcal{AC}(\pi')) = \mathcal{AC}(\theta(\pi')),$$

where θ(associated cycle) means the theta lifting of nilpotent orbits in the stable range. We also obtained a K-type formula for $\theta(\pi')$ in terms of the branching coefficient of classical groups; the associated nilpotent orbit is realized as a quotient of a minimal nilpotent orbit of a lager group. The K-type formula is not new though, since $\theta(\pi')$ is a derived functor module. However, our K-type formula is not a variant of Blattner's one, and we believe ours has some advantage.

Also, we have given a brief survey on the associated cycles of the unitary lowest weight representations in the terminology of classical invariant theory ([1]). This idea is crucial for the investigation of the theta lifting of the lowest weight representations explained above.

The talk is based on the joint research ([2], [3], [4]) with Chen-bo Zhu (National University of Singapore) and Hiroyuki Ochiai (Nagoya University).

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Quantum Chaos and Cohomology of Arithmetic Groups

Joachim Hilgert
(joint work with A. Deitmar)

Our work [1] is motivated by the following problem: given a classical system (symplectic manifold plus Hamiltonian function) and a quantization of this system (Hilbert space plus a self adjoint operator), can one detect from the quantum system whether the classical system shows chaotic behavior (e.g. ergodic or hyperbolic behavior)? For the modular surface and its geodesic flow (so that a suitable quantization is given by the corresponding L^2-space together with the Laplace-Beltrami operator Δ) Lewis and Zagier [2] have constructed a natural correspondence between Maass cusp forms (which are eigenfunctions of Δ) and holomorphic functions $\psi\colon \mathbb{C} \setminus \mathbb{R}^- \to \mathbb{C}$ satisfying a three term functional equation (called the Lewis equation) which has a natural interpretation in terms of the classical system. So far one has this correspondence only for this surfaces, but it is expected that it can be extended to coverings or even more general locally symmetric spaces of finite volume.

The Lewis equation admits a cohomological interpretation which suggests a starting point for generalizations. On the other hand Maass cusp forms can be defined in terms of representation theory and correspond to Γ-invariant vectors in principal series representations π_s of $\mathrm{PSL}(2,\mathbb{R})$, which leads to an interpretation of the dimension of the space of Maass cusp forms as multiplicities $N_\Gamma(\pi_s)$ of π_s in $L^2(\Gamma\backslash G)$.

Our main theorem is the following multiplicity formula for split semisimple Lie groups with arithmetic torsion free subgroups: If π is any irreducible unitary principal series representation and r, d the rank, respectively the dimension of the non-compact Riemannian symmetric space associated with G, then

$$N_\Gamma(\pi) = \sum_{j\geq 0} (-1)^{j+r} \binom{j}{r} \dim H^{N-j}_{\mathrm{cusp}}(\Gamma, \pi^\omega).$$

In order to prove this formula establish a functorial isomorphism

$$H^j(\mathfrak{g}, KF\hat{\otimes}V^{\max} \to \mathrm{Ext}^j_{\mathfrak{g},K}(\tilde{V}, F)$$

for Harish-Chandra modules V (then $V^{\max}$ is the maximal globalization and $\tilde{V}$ is the dual Harisch-Chandra module) and smooth G-representations F, as well as a Poincaré duality

$$H^j_{\text{cusp}}(\Gamma, V^{\max}) \cong H^{N-j}_{\text{cusp}}(\Gamma, \tilde{V}^{\min}),$$

where $\tilde{V}^{\min}$ is the minimal globalization of $\tilde{V}$. As a corollary we derive

Theorem: Let Γ be a Fuchsian group of finite covolume and $s \in \mathbb{R}$. Then $N_\Gamma(\pi_s) = \dim H^1_{\text{cusp}}(\Gamma, \pi_s^\omega)$, where π_s^ω is the G-module of analytic vectors in the representation space of π_s.

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Global deformations of the Virasoro algebra

Alice Fialowski

This talk is based on a joint work with Martin Schlichenmaier (see [4]).

Introduction. Deformation is one of the tools to study a specific object, by deforming it into some families of "similar" structure objects. Another question related to deformation: Can we equip the set of nonequivalent deformations with the structure of a topological or maybe geometric space? In other words, does there exist a moduli space for these structures? If so, then for a fixed object its deformations should reflect the local structure of the moduli space at the point corresponding to this object.

There is a lot of confusion in the literature in the notion of a deformation. Several different (inequivalent) approaches exist. May aim now is to clarify the difference between deformations of geometric origin and so-called formal deformations. Formal deformation theory has the advantage of using cohomology. It is also complete in the sense that under some natural cohomology assumptions there exists a versal formal deformation which induces all other deformations. Formal deformations are deformations with a complete local algebra base. A deformation with a commutative (non-local) algebra base gives a much richer picture of deformation families, depending on the augmentation of the base algebra. If we identify the base of deformation – which is a commutative algebra of functions – with a smooth manifold, an augmentation corresponds to choosing a point on the manifold. So choosing different points should in general lead to different deformation situations. I will show in the case of the Witt and Virasoro algebra that – in the case of infinite dimensional Lie algebras – there is no tight relation between global formal deformations.

1. Deformations. Let $\mathcal{L}$ be a Lie algebra.

i) Intuitively: One-parameter family $\mathcal{L}_t$ of Lie algebras with bracket $\mu_t = \mu_0 + t\phi_1 + t^2\phi_2 + \dots$.

ii) Global deformations: Consider a deformation $\mathcal{L}_t$ not as a family of Lie algebras, but as a Lie algebra over the algebra $\mathbb{K}[[t]]$. Call it the *base* of the deformation. The natural generalization is to allow more parameters, or to take in general a commutative algebra A over $\mathbb{K}$ with identity as base of a deformation. Take such an A over $\mathbb{K}$ of char 0 with an augmentation $\varepsilon : A \to \mathbb{K}$ and $m = \operatorname{Ker}\varepsilon$ maximal ideal.

Definition. A *global deformation* λ of $\mathcal{L}$ *with base* (A, m) is a Lie A-algebra structure on $A \otimes_{\mathbb{K}} \mathcal{L}$ with $[\ ,\ ]_\lambda$ such that $\varepsilon \otimes \mathrm{id} : A \otimes \mathcal{L} \to \mathbb{K} \otimes \mathcal{L} = \mathcal{L}$ is a Lie algebra homomorphism.

A deformation is called *trivial* if $A \otimes_{\mathbb{K}} \mathcal{L}$ carries the trivially extended Lie structure, i.e. $[1 \otimes x, 1 \otimes y]_\lambda = 1 \otimes [x, y]$. Two deformations of a Lie algebra $\mathcal{L}$ with the same base A are called *equivalent* if there exists a Lie algebra isomorphism between the two copies of $A \otimes \mathcal{L}$ with the two Lie algebra structures, compatible with $\varepsilon \otimes \mathrm{id}$. We say that the deformation is *local* if A is a local $\mathbb{K}$-algebra with unique maximal ideal $m_A = \operatorname{Ker}\varepsilon$. In case that in addition, $m_A^2 > 0$, the deformation is called *infinitesimal*.

iii) We call a deformation *formal*, if its base is a complete local algebra (with a unique maximal ideal) (see [1]).

Proposition (see [3])**.** *If* $\dim \mathrm{H}^2(\mathcal{L}, \mathcal{L}) < \infty$*, there exists a universal infinitesimal deformation* $\eta_{\mathcal{L}}$ *of* $\mathcal{L}$ *with base* $B = \mathbb{K} \oplus \mathrm{H}^2(\mathcal{L}, \mathcal{L})'$.

This means that for any infinitesimal deformation λ of the Lie algebra $\mathcal{L}$ with finite-dimensional (local) algebra base A there exists a unique homomorphism $\phi : \mathbb{K} \oplus \mathrm{H}^2(\mathcal{L}, \mathcal{L})' \to A$ such that λ is equivalent to the push-out $\phi_* \eta_{\mathcal{L}}$.

Definition ([1])**.** A formal deformation η of $\mathcal{L}$ parametrized by a complete local algebra B is called *versal* if for any deformation λ, parametrized by (A, m_A), there exists $f : B \to A$ morphism such that the push-out

1) $f_* \eta$ is equivalent to λ.

2) If A satisfies ${m_A}^2 = 0$, then f is unique.

Theorem. *Assume* $\mathrm{H}^2(\mathcal{L}, \mathcal{L})$ *is finite dimensional.*

a) ([1]) *There exists a versal formal deformation of* $\mathcal{L}$.

b) ([3]) *The base of the versal deformation is formally embedded into* $\mathrm{H}^2(\mathcal{L}, \mathcal{L})$*, i.e. it can be described in* $\mathrm{H}^2(\mathcal{L}, \mathcal{L})$ *by a finite system of formal equation.*

Corollary. $\mathrm{H}^2(\mathcal{L}, \mathcal{L}) = \{0\}$ *implies that* $\mathcal{L}$ *is formally rigid.*

Theorem ([2])**.** *The Witt and Virasoro algebra is formally rigid.*

2. Krichever–Novikov algebras. They are generalizations of the Virasoro and all its related algebras. Let M be a compact Riemann surface of genus g, or a smooth projective curve over $\mathbb{C}$. Let $I = \{P\}$ and $O = \{Q\}$ be distinct points ("marked points") on the curve. Denote $A = I \cup O$ as a set. Denote by $\mathcal{L}$ the Lie algebra consisting of those meromorphic sections of the holomorphic tangent line bundle which are holomorphic outside of A, equipped with the Lie bracket of vector field. Call them Krichever–Novikov algebras. For the Riemann sphere ($g = 0$) with quasi-global coordinate z, $I = \{0\}$, $O = \{\infty\}$, the introduced algebra is the Witt algebra. The Witt and Virasoro algebras are graded, but these Krichever–Novikov algebras are only almost graded, as was observed by Krichever–Novikov in the two-point case [5] and generalized by Schlichenmaier [6].

We consider the genus one case, i.e., the case of one-dimensional complex tori, or, equivalently the elliptic curve case. Consider now two marked points. One marking we always put to $\infty = (0 : 1 : 0)$, and the other one to the affine coordinate $(e, 0)$. Set

$$B := \{(e_1, e_2, e_3) \in \mathbb{C}^3 \mid e_1 + e_2 + e_3 = 0,\ e_i \neq e_j \text{ for } i \neq j\}.$$

In $B \times \mathbb{P}^2$ we consider the family of elliptic curves $\mathcal{E}$ over B defined via $Y^2Z = 4(X - e_1Z)(X - e_2Z)(X - e_3Z)$. Consider the complex lines in $\mathbb{C}^2$:

$$D_s := \{(e_1, e_2) \in \mathbb{C}^2 \mid e_2 = s \cdot e_1\},\ s \in \mathbb{C}, \quad D_\infty := \{(0, e_2) \in \mathbb{C}^2\}.$$

Then B is isomorphic to $\mathbb{C}^2 \setminus (D_1 \cup D_{-\frac{1}{2}} \cup D_{-2})$.

Theorem ([7]). *For any elliptic curve $E_{(e_1,e_2)}$ over $(e_1, e_2) \in \mathbb{C}^2 \setminus (D_1 \cup D_{-1/2} \cup D_{-2})$ the Lie algebra $\mathcal{L}^{(e_1,e_2)}$ of vector fields on $E_{e_1,e_2)}$ has a basis $\{V_n, n \in \mathbb{Z}\}$ such that the Lie algebra structure is given as*

$$(*) \qquad [V_n, V_m] = \begin{cases} (m-n)V_{n+m}, & n, m \text{ odd}, \\ (m-n)(V_{n+m} + 3e_1V_{n+m-2} & \\ \quad +(e_1 - e_2)(e_1 - e_3)V_{n+m-4}), & n, m \text{ even}, \\ (m-n)V_{n+m} + (m-n-1)3e_1V_{n+m-2} & \\ \quad +(m-n-2)(e_1 - e_2)(e_1 - e_3)V_{n+m-4}, & n \text{ odd, } m \text{ even}. \end{cases}$$

These algebras make sense also for the points $(e_1, e_2) \in D_1 \cup D_{-\frac{1}{2}} \cup D_{-2}$. Altogether this defines a 2-dimensional family of Lie algebras parametrized over $\mathbb{C}^2$. In particular, for $(e_1, e_2) = 0$ we get the Witt algebra.

Now consider the family of algebras obtained by taking as base variety the line D_s (for an s). We get that for fixed s in all cases the algebras will be isomorphic above every point in D_s as long as we are not above $(0, 0)$.

Proposition. *For $(e_1, e_2) \neq (0, 0)$ the algebras $\mathcal{L}^{(e_1,e_2)}$ are* not *isomorphic to $\mathcal{W}$.*

In particular, we obtain a family of algebras over the base D_s, which is always the affine line. In this family, the algebra over the point $(0, 0)$ is the Witt algebra and the isomorphy type above all other points will be the same but different from this special Witt element. We obtain the following

Theorem. *For every $s \in \mathbb{C} \cup \{\infty\}$ the families of Lie algebras defined by $(*)$ define global deformations $\mathcal{W}_t^{(s)}$ of $\mathcal{W}$ over the affine line $\mathbb{C}[t]$. Here t corresponds to the parameter e_1 and e_2 respectively. The Lie algebra over $t = 0$ corresponds* *always to the Witt algebra, the algebras above $t \neq 0$ belong (if s is fixed) to the same isomorphy class, but are not isomorphic to $\mathcal{W}$.*

Remark. It is easy to incorporate a central term defined by a local cocycle and easy to show that the centrally extended algebras have the same properties.

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Direct limits of Lie groups

Helge Glöckner

1. Existing methods. Let $G_1 \subseteq G_2 \subseteq \cdots$ be an ascending sequence of finite-dimensional real Lie groups, such that the inclusion maps are smooth homomorphisms. Then $G := \bigcup_{n\in\mathbb{N}} G_n$ is a group in a natural way, and it becomes a topological group when equipped with the final topology with respect to the inclusion maps $G_n \to G$, the so-called *DL-topology* ([1], [11]). Provided certain technical conditions are satisfied (ensuring in particular that $\exp_G := \varinjlim \exp_{G_n} : \varinjlim L(G_n) \to \varinjlim G_n = G$ is a local homeomorphism at 0), the map $\exp_G$ and its translates can be used as charts which make G a (usually infinite-dimensional) Lie group (see [9] and subsequent work by the same authors). It is also known that every Lie subalgebra of $\mathfrak{gl}_\infty(\mathbb{R}) := \varinjlim \mathfrak{gl}_n(\mathbb{R})$ integrates to a subgroup of $\mathrm{GL}_\infty(\mathbb{R}) := \varinjlim \mathrm{GL}_n(\mathbb{R})$ [6]; this facilitates an alternative construction of a Lie group structure on various direct limits of linear Lie groups. However, neither of these methods is general enough to tackle arbitrary direct limits of Lie groups. In particular, examples show that $\exp_G$ need not be injective on any 0-neighbourhood [1], whence a general construction of a Lie group structure on $G = \bigcup_n G_n$ cannot make use of $\exp_G$.

2. A new construction principle. In [1], a Lie group structure on $G = \bigcup_n G_n$ was constructed in the case where the inclusion maps are embeddings (strict direct systems). Later, the strictness condition could be removed [2]. In [2], direct limits of Lie groups are discussed as special cases of direct limits of direct sequences

$M_1 \subseteq M_2 \subseteq \cdots$ of finite-dimensional smooth manifolds and injective immersions. To make $M := \bigcup_n M_n$ a smooth manifold, the idea is to start with a chart ϕ_1 of some M_n (say M_1) and then to use tubular neighbourhoods to extend ϕ_n already defined (or its restriction to a slightly smaller open set) to a chart of M_{n+1}. Then $\varinjlim \phi_n$ is a chart for M. It can be shown that M is smoothly paracompact [2]. Furthermore (see [2]), the direct limit groups G are regular Lie groups in the sense of convenient differential calculus [6] (this is easy) and also regular Lie groups in Milnor's sense [8] (this is much harder to prove). If all manifolds (or Lie groups) and all bonding maps are real or complex analytic, then the direct limit manifolds constructed in [2] are real analytic in the sense of convenient differential calculus, resp., complex analytic.

3. Lie theory for direct limit groups. Despite the fact that $\exp_G$ need not be well-behaved, all of the basic constructions of finite-dimensional Lie theory can be pushed to the case of direct limit groups $G = \bigcup_n G_n$. Thus, subgroups and Hausdorff quotient groups of G are Lie groups, a universal complexification $G_{\mathbb{C}}$ exists, subalgebras of $L(G)$ integrate to analytic subgroups, and Lie algebra homomorphisms integrate to smooth homomorphisms in the expected way. Furthermore, every locally finite real or complex Lie algebra of countable dimension is enlargible, i.e., it is the Lie algebra of a regular Lie group [2]. Such Lie algebras have been studied by Bahturin, Baranov, Benkart, Dimitrov, Neeb, Penkov, Strade, Stumme, and Zalesskii. If $H \subseteq G$ is a closed subgroup, then H is a smooth submanifold of G, and in fact a conveniently real analytic (c^ω-) submanifold, under mild additional conditions [2]. Furthermore, the homogeneous space G/H can be given a c^ω-manifold structure which makes $\pi\colon G \to G/H$ a smooth principal bundle (and a c^ω-principal bundle under additional conditions), [2]. Similar results are available for complex Lie groups [2]. Special cases of complexifications and homogeneous spaces have already been used in [10], in the context of a Bott-Borel-Weil theorem for direct limit groups.

4. Direct limits of infinite-dimensional Lie groups. The situation becomes more complicated if the G_n's are infinite-dimensional Lie groups. Let us assume that a direct limit $\phi := \varinjlim \phi_n$ of compatible charts is defined on some open (or c^∞-open) subset of the locally convex direct limit $\varinjlim L(G_n)$. Provided $\varinjlim L(G_n)$ is regular (viz. it is Hausdorff, and each bounded subset is contained and bounded in some $L(G_n)$), then it is straightforward to make $G = \bigcup_n G_n$ a (possibly not smoothly Hausdorff) Lie group in the sense of convenient differential calculus [4], whose group multiplication however need not be continuous (cf. [11]). All Lie groups of relevance are Lie groups in a stronger sense (as in Milnor [8]), based on a notion of smooth maps which are, in particular, continuous (Keller's C^∞_c-maps). Pathological examples show that, even if ϕ is a global chart, it need not make $G = \bigcup_n G_n$ a Milnor-Lie group, [4]. But what happens for the examples encountered in practice?

5. Discussion of the main examples. Given a σ-compact smooth manifold M of finite dimension, the group $\mathrm{Diff}_c(M)$ of compactly supported smooth diffeomorphisms of M is a Lie group in Milnor's sense (see [7] or [5], where also regularity of $\mathrm{Diff}_c(M)$ in Milnor's sense is proved in detail). It is a union $\mathrm{Diff}_c(M) = \bigcup_K \mathrm{Diff}_K(M)$ of the Fréchet-Lie groups $\mathrm{Diff}_K(M)$ of diffeomorphisms supported in a given compact subset $K \subseteq M$. Because the DL-topology does not make $\mathrm{Diff}_c(M)$ a topological group [11], the DL-topology is strictly finer than the topology on the Lie group $\mathrm{Diff}_c(M)$. Hence, there exists a discontinuous map on $\mathrm{Diff}_c(M)$ which is continuous on $\mathrm{Diff}_K(M)$ for each K. There even exists a discontinuous map on $\mathrm{Diff}_K(M)$ which is smooth on each $\mathrm{Diff}_K(M)$, whence $\mathrm{Diff}_c(M) \neq \varinjlim \mathrm{Diff}_K(M)$ as a smooth manifold [4]. However, homomorphisms on $\mathrm{Diff}_c(M)$ are smooth (resp., continuous) if and only if they are so on each $\mathrm{Diff}_K(M)$, [4]. The situation is similar for test function groups $C^\infty_c(M,G)$ with values in a Lie group G. Thus $\mathrm{Diff}_c(M) = \varinjlim \mathrm{Diff}_K(M)$ and $C^\infty_c(M,G) = \varinjlim C^\infty_K(M,G)$ holds or does not hold, in the following categories (see [4]):

	$C^\infty_c(M,G)$	$\mathrm{Diff}_c(M)$
Lie groups	yes	yes
topological groups	yes	yes
smooth manifolds	no	no
topological spaces	no	no

6. Smooth homomorphisms vs. continuous homomorphisms. The continuity and smoothness questions just analyzed are related to the general (open) problem (due to Milnor) whether every continuous homomorphism between infinite-dimensional Lie groups is smooth. Some progress concerning this problem has been made recently: Every *Hölder* continuous homomorphism between Milnor-Lie groups is smooth [3], and Lip^0-homomorphisms between Lie groups in the sense of convenient differential calculus are smooth in the convenient sense (the author, work in progress).

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Flag manifolds and cycles
Gregor Fels

Let G be a complex semisimple Lie group and $Q \subset G$ a parabolic subgroup. Let $S \subset G$ be a (connected) real form of G. Let $\mathfrak{s} \subset \mathfrak{g} = \mathfrak{s}^{\mathbb{C}}$ denote the corresponding Lie algebras. Fix a Cartan decomposition $\mathfrak{s} = \mathfrak{k} \oplus \mathfrak{p}$ and let $\mathfrak{g} = \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{p}^{\mathbb{C}}$ be its complexification. Finally, let $K \subset S$ denote the corresponding maximal compact subgroup and let $K^{\mathbb{C}} \subset G$ be its complexification. In order to avoid some awkward case by case distinctions we assume that G is simple. All the result below can be easily generalized for semisimple G.

Let $X := G/Q$ be a flag manifold. The orbit structure of the canonical action $S \times X \to X$ by left translations is well understood, see [Wo1]. Since there are only finitely many Sfrm-e–orbits in X we conclude that open orbits exist. Any open Sfrm-e–orbit is called a flag domain.
Every flag domains $D = S{\cdot}x$ contains a unique compact $K^{\mathbb{C}}$frm-e–orbit C_D. Such orbit has the property $C_D = K^{\mathbb{C}}{\cdot}x = K{\cdot}x$. This is a special case of a more general fact: There is a natural duality between the Sfrm-e–orbits and the $K^{\mathbb{C}}$frm-e–orbits in X, and an Sfrm-e–orbit $\mathbf{s}$ and an $K^{\mathbb{C}}$frm-e-orbit $\mathbf{k}$ are said to be dual if the intersection $\mathbf{s} \cap \mathbf{k}$ is a single Kfrm-e–orbit. see [Mat], [MUV], [BrLo].
Every compact $K^{\mathbb{C}}$frm-e–orbit C_D defines a cycle $1{\cdot}C_D$ in X i.e., a point in the Barlet cycle space. The Barlet cycle space $\mathfrak{C}(X)$ provides a universal family parameterizing all cycles in X. The construction of the Barlet space $\mathfrak{C}(Z)$ can be given for an arbitrary complex space Z, see [Bar] for the details. From the point of view of group actions, a natural family can be defined as follows ([WeWo]). For a given compact $K^{\mathbb{C}}$frm-e–orbit $C = C_D$, consider $\widetilde{\mathcal{M}}_D := \{g \in G \mid gC \subset D\}$.
Notice that the stabilizer $G_{[C]}$ of C acts freely and properly on the right on this set, and that the quotient

$$\mathcal{M}_D := (\widetilde{\mathcal{M}}_D/G_{[C]})^{\circ}$$

can be identified with a domain in the complex homogenous space $G/G_{[C]}$. Observe that this space parameterizes the (connected component) of the family of submanifolds of D which are obtained by moving the base manifold C by elements of $g \in G$ such that $g(C) \subset D$. We refer to such $\mathcal{M}_D$ as the Wolf parameter space.
The analysis of the quotient $G/G_{[C]}$ shows that the following cases occur:

- $G/G_{[C]} = \{\text{pt}\}$ in the rare case when a non-compact real form S acts transitively on X
- $G/G_{[C]}$ is a compact Hermitian space Y. This happens only if S is of Hermitian type and certain cycles $C_D \subset G/Q$

• $G/G_{[C]}$ is the affine symmetric space $G/N_G(\mathfrak{k}^{\mathbb{C}})$.

Our first main result is the description of the Wolf parameter spaces $\mathcal{M}_D$ for all S, X and the corresponding flag domains D. In the particular case when S is of Hermitian type, the structure of $\mathcal{M}_D$ was determined in [WZ]: In this case $\mathcal{M}_D \cong \Delta$ or $\mathcal{M}_D \cong \Delta \times \overline{\Delta}$ where Δ denotes the bounded symmetric domain such that $\mathrm{Aut}^\circ(\Delta) = S/Z(S)$.

We deal only with the case where S is not of Hermitian case. Let $H := N_G(\mathfrak{k}^{\mathbb{C}}) = G_{[C_D]}$. It turns out that

Theorem 1. *Let a (non-Hermitian) real form S be fixed. For arbitrary X and flag domain $D \subset X$ all domains $\mathcal{M}_{D_C} \subset G/H$ coincide.* The domains $\mathcal{M}_{D_C}$ can

be also described in a more explicite way: Fix a maximal Abelian subalgebra $\mathfrak{a} \subset \mathfrak{p}$

and an Iwasawa decomposition $\mathfrak{s} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$. Fix a Borel subgroup $B \subset G$ such that $\mathfrak{b} \supset \mathfrak{a} \oplus \mathfrak{n}$. It should be noted that $B\cdot[H]$ is open in G/H and its complement consists of $\dim \mathfrak{a}$ irreducible Bfrm-e–stable hypersurfaces: $G/H \setminus B\cdot[H] = \mathcal{H}_1 \cup \cdots \cup \mathcal{H}_r$. For any Bfrm-e–stable hypersurface $\mathcal{H}$ define the set

$$\Omega_{\mathcal{H}} := \big(G/H \setminus \bigcup_{s\in S} s\mathcal{H}\big)^\circ = \big(G/H \setminus \bigcup_{k\in K} k\mathcal{H}\big)^\circ$$

where $(\cdots)^\circ$ denote the connected component containing $[H]$. This set is open and is called the hypersurface domain, associated with $\mathcal{H}$.

Theorem 2. *For an arbitrary but fixed (non-Hermitian) S, any flag domain $D \subset X$ and arbitrary Bfrm-e–stable divisor $\mathcal{H} \subset G/H$ we have*

$$\mathcal{M}_D = \Omega_{\mathcal{H}} = S\cdot \exp i\omega_{AG}\cdot[H]$$

where $\omega_{AG} := \{X \in \mathfrak{a} \mid |\lambda(X)| < \pi/2$ for all $\lambda \in \Phi(\mathfrak{a})\}$. Here, $\Phi(\mathfrak{a})$ denotes the restricted root system of $\mathfrak{s}$ with respect to $\mathfrak{a}$.
All above domains are Kobayashi hyperbolic.

See [FeHu], [HuWo].

Remark. The open set $S\cdot \exp i\omega_{AG}\cdot[H] \subset G/H$ is also called the Akhiezer-Gindikin domain, see [AG]. Note that H is disconnected and $K^{\mathbb{C}} = H^\circ$.
The covering map $G/K^{\mathbb{C}} \to G/H$ maps biholomorphically $S\cdot \exp i\omega_{AG}\cdot[K^{\mathbb{C}}]$ onto $S\cdot \exp i\omega_{AG}\cdot[H]$. Furthermore, being interested in local properties of various cycle spaces, we do not need to distinguish between $H = G_{[C]}$ and $K^{\mathbb{C}} = H^\circ$.

As already mentioned, instead of moving the reference cycles C_D by elements of a given transformation group G one can also consider the universal family of cycles, i.e., the component of the Barlet cycle space $\mathfrak{C}(X)$ which contains C_D. Such a universal family depends only on the complex geometry of the ambient space and the embedding $C_D \hookrightarrow X$. A compact $K^{\mathbb{C}}$frm-e–orbit C can be now considered as a point $[C] \in \mathfrak{C}(D) = \mathfrak{C}$, and $\mathcal{M}_D$ is just a subset of $\mathfrak{C}$. Therefore one naturally asks if $\mathcal{M}_D = \mathfrak{C}(D)$ or if $\mathcal{M}_D$ is at least open in $\mathfrak{C}$.

In order to "see" cycles in the proximity of the given C it is necessary to compute the full Zariski tangent space $T_{[C]}\mathfrak{C}$ at a point $[C]$. In general, the Barlet spaces $\mathfrak{C}$ are singular and in principle, the point $[C] \in \mathfrak{C}$ might be singular. Note that we have the canonical subspace $T_{[C]}(G{\cdot}[C])$ of $T_{[C]}\mathfrak{C}$, consisting of vectors tangent to the orbit $G{\cdot}[C]$.

Our first result here is that for *certain* real forms $S \subset G$ the tangent spaces to $\mathfrak{C}$ computed at all compact $K^{\mathbb{C}}$frm-e–orbits C and for all G-flags X the spaces $T_o(G{\cdot}[C])$ and $T_{[C]}\mathfrak{C}$ coincide. In particular, $\mathcal{M}_D$ is open in $\mathfrak{C}(X)$.

On the other hand, there are real forms S and flag manifolds X in which there are situations which do not arise by moving the base cycle by elements of $\mathrm{Aut}(X)$: There exist real forms and compact orbits $C \subset X$ (we give a precise list below) such that $\dim T_{[C]}\mathfrak{C} > \dim G/G_{[C]}$. In such a case we compute in detail the isotropy representation $K^{\mathbb{C}} \times T_{[C]}\mathfrak{C} \to T_{[C]}\mathfrak{C}$. It is actually quite difficult to obtain precise quantitative results of this type, and a substantial part of our work consists in developing effective methods for computing certain cohomology groups which are necessary for our purposes.

The calculations are carried out mostly for the full flag $X = G/B$. It should be noted that in this case $\mathrm{Aut}^{\circ}(X) = G/Z(G)$.

Theorem 3. *In all cases the Barlet space $\mathfrak{C}(G/B)$ is smooth at C_D.*

Note that for G simple and $S \subset G$ a real form, all $\mathfrak{k}^{\mathbb{C}}$frm-e–modules $\mathfrak{p}^{\mathbb{C}}$ in the complexified Cartan decomposition of $\mathfrak{s}$ are irreducible if S is not of Hermitian type and sum of two irreducible submodules $\mathfrak{p}^{\mathbb{C}} = (\mathfrak{p}^{\mathbb{C}})^{+} \oplus (\mathfrak{p}^{\mathbb{C}})^{-}$ if S is of Hermitian type. Further, for every complex group H of the classical type A–D let $H \hookrightarrow \mathrm{GL}(V^{H}_{\mathrm{std}})$ denote the standard representation. It turns out that the isotropy groups in all cases listed below are of classical type.

Theorem 4.

i) *For all real forms $\mathfrak{s}$ listed below, there exist compact $K^{\mathbb{C}}$frm-e–orbits $C \subset G/B = X$, such that the Zariski tangent space $T_{[C]}\mathfrak{C}(X)$ is bigger than $T_{[C]}(G/G_{[C]})$. $T_{[C]}\mathfrak{C} = T_{[C]}(G{\cdot}[C])$. The real forms listed below are also the only ones with this property:*
   (1) $\mathfrak{so}(2p, 2q+1)$ *for* $p \geq 2$,
   (2) $\mathfrak{so}(2p+1, 2q+1)$, *for* $p, q \geq 1$
   (3) $\mathfrak{sp}_n(\mathbb{R})$ *for* $n \geq 3$,
   (4) G_2
   (5) $\mathfrak{sl}_3(\mathbb{R})$.

ii) *At the same time, for all real forms except $\mathfrak{sl}_3(\mathbb{R})$ there exist compact $K^{\mathbb{C}}$frm-e–orbits $C' \subset G/B$, such that $T_{[C']}\mathfrak{C}(X) = T_{[C']}(G{\cdot}[C'])$.*

iii) *For those compact $K^{\mathbb{C}}$frm-e–orbits $C \subset G/B$ with the property as in i) the tangent space $T_{[C]}\mathfrak{C}(X)$ has the following decomposition as a $K^{\mathbb{C}}$frm-e–module:*

$\mathfrak{s} = \mathfrak{so}(2p, 2q+1)$	$T_{[C]}\mathfrak{C} = \mathfrak{p}^{\mathbb{C}} \oplus V_{\mathrm{std}}^{\mathrm{SO}_{2p}}$
$\mathfrak{s} = \mathfrak{so}(2p+1, 2q+1)$	$T_{[C]}\mathfrak{C} = \mathfrak{p}^{\mathbb{C}} \oplus V_{\mathrm{std}}^{\mathrm{SO}_{2p+1}}$ *or* $T_{[C]}\mathfrak{C} = \mathfrak{p}^{\mathbb{C}} \oplus V_{\mathrm{std}}^{\mathrm{SO}_{2q+1}}$
$\mathfrak{s} = \mathfrak{sp}_n(\mathbb{R})$	$T_{[C]}\mathfrak{C} = (\mathfrak{p}^{\mathbb{C}})^+ \oplus \bigwedge^2 V_{\mathrm{std}}^{\mathrm{GL}_n}$ *or* $T_{[C]}\mathfrak{C} = (\mathfrak{p}^{\mathbb{C}})^- \oplus \bigwedge^2 (V_{\mathrm{std}}^{\mathrm{GL}_n})^*$
$\mathfrak{s} = \mathbf{G}_2$	$T_{[C]}\mathfrak{C} = \mathfrak{p}^{\mathbb{C}} \oplus V_{\mathrm{std}}^{\mathrm{SO}_4}$
$\mathfrak{s} = \mathfrak{sl}_3(\mathbb{R})$	$T_{[C]}\mathfrak{C} = \mathfrak{p}^{\mathbb{C}} \oplus V_{\mathrm{std}}^{\mathrm{SO}_3}$

See [Fe] for the proofs and further details.

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Berezin transform on root systems of type BC

Genkai Zhang

In the present talk we present our recent result on Berezin transform on root systems with general multiplicities. The Berezn transform on symmetric domains arises when one studies the branching of holomorphic representation on a Hermitian symmetric space G/K of a semisimple Lie group G under a symmetric subgroup H with the corresponding symmetric space H/L being a real form of G/K. More precisely, consider the restriction map R of a scalar holomorphic discrete series $\mathcal{H}_\nu$ (and its analytic continuation) realized as a Hilbert space of holomorphic functions on G/K to the real form H/L. The map $B_\nu = RR^*$ on $L^2(H/L)$ is then the Berezin transform. It is H-invariant, and is bounded on $L^2(H/L)$ for larger parameter of ν. The spectral symbol of B_ν has been computed by Unterberger-Upmeier [3], Zhang [6] [5] van Dijk and Pevsner [1] and have found several applications [4]. In the present work we consider a general root system of type BC with general positive multiplicity. The Berezin transform can be defined as an integral operator whose kernel is defined by a series. We find the spectral symbol of the Berezin transform and find a Bernstein-Sato type formula for the integral kernel. The precise results are summarized below.

Let $\mathfrak{a} = \mathbb{R}^r$ be an Euclidean space with inner product $(\cdot,\cdot)$ and let $R \subset \mathfrak{a}^*$ be a root system of type BC. We fix an orthogonal basis γ_j, $j = 1, \ldots, r$ of $\mathfrak{a}^*$ so that $R = \{\frac{\gamma_j}{2}; j = 1, \cdots, r\} \cup \{\gamma_j; j = 1, \cdots, r\} \cup \{\frac{\gamma_j \pm \gamma_k}{2}; j \neq k = 1, \cdots, r\}$ and let $k = (k_1, k_2, k_3)$ be the root multiplicity with k_1, k_2 and k_3 the multiplicities of the respective subsets of R. We assume that $k_1 + k_2 > 0$ and $k_3 > 0$.

Let $\{\xi_j\}$ be the basis of $\mathfrak{a}$ dual to $\frac{\gamma_j}{2}$, $j = 1, \ldots, r$, i.e., $\frac{\gamma_j}{2}(\xi_k) = \delta_{jk}$. A function $f(x)$ on $\mathfrak{a}^{\mathbb{C}}$ will be identified with $f(x_1, \cdots, x_r)$, for $x = x_1\xi_1 + \cdots + x_r\xi_r$. Let $D_j = D_{\xi_j}$ be be the Cherednik operators and let ϕ_λ be Heckman-Opdam the spherical function. Consider the function

$$f_\nu(x) = \prod_{j=1}^{r} \cosh(x_j)^{-2\nu}.$$

The integral kernel $B(x, y)$ of the Berezin transform is given by $B(x, 0) = f_\nu(x)$ and by an infinite series with using the Jack symmetric polynomials. Its spectral symbol is determined by the integral

$$b_\nu(\lambda) = \widetilde{f_\nu}(\lambda) = \int_{\mathfrak{a}} f_\nu(x)\phi_\lambda(x)d\mu_\Sigma(x)$$

where $d\mu_\Sigma(x)$ is the invariant measure for the root system Σ (which corresponds to the radial A-part of the Riemannian measure in the case of symmetric space $H/L = LA \cdot 0$).

We prove first a Bernstein-Sato type formula using the Cherednik operator.

Theorem 1. *(Bernstein-Sato type formula) The following formula holds*

$$\prod_{j=1}^{r}(D_j^2 - (-\nu/2 + \rho_1)^2))f_\nu = \prod_{j=1}^{r}(-\nu/2 + k_3(j-1))(1 + \nu/2 - k_2 - k_3(r-j))f_{\nu+1}$$

In proving the theorem we also find some interesting commutation relation for the Hecke algebra elements and multiplication operators by polynomials of e_j^x.

We can then find the spectral symbol.

Theorem 2. *The spherical transform of f_ν is given by*

$$b_\nu(\lambda) = c_\delta \prod_{j=1}^{r} \prod_{\varepsilon=\pm 1} \Gamma(\nu - \frac{p-1}{2} + \varepsilon\lambda_j)$$

The result has also some applications to orthogonal polynomials, the details will appear later.

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General Differential Calculus and General Lie Theory

Wolfgang Bertram

In joint work with H. Glöckner and K.-H. Neeb [1], a simple and at the same time very general approach to differential calculus, manifolds and Lie groups is proposed which not only works in arbitrary dimension over the real and complex numbers, but more generally for arbitrary topological modules over (commutative) base rings k having a dense group of invertible elements (in particular, over all non-discrete topological fields). All notions and results from differential geometry and Lie theory that are essentially algebraic in nature continue to make sense in this general framework – one may call these parts of the theory "general differential geometry" and "general Lie theory".

In our talk we present a basic result of this theory which in a way provides a rigorous justification of the use of "infinitesimals" in differential geometry (cf. [3]): if M is a manifold over k, then the tangent bundle TM is, in a natural way, a manifold over the ring of dual numbers $k[\epsilon] = k \oplus \epsilon k \cong k[x]/(x^2)$ (with relation $\epsilon^2 = 0$), and tangent maps are smooth over $k[\epsilon]$; thus the tangent functor really is a functor of scalar extension from k to dual numbers over k. It immediately follows that the iterated tangent bundles T^nM are manifolds over the ring $T^nk := k[\epsilon_1, \ldots, \epsilon_n]$ and that the "jet bundles" $J^nM = (T^nM)^{\Sigma_n}$ (the subbundle fixed under the canonical action of the permutation group Σ_n on T^nM) are manifolds over the ring $J^nk := (T^nk)^{\Sigma_n}$. Likewise, if G is a Lie group over k, then T^nG is a Lie group over T^nk and J^nG is a Lie group over J^nk. Another approach to infinitesimals has been proposed by A. Weil in 1953 and lead to various concepts such as the "Weil-functors" defined in the book "Natural Operations in Differential Geometry" by I. Kolář, P. Michor and J. Slovák (Springer-Verlag 1993) or the theory of "smooth toposes" and "synthetic differential geometry" (see the book

"Models for Smooth Infinitesimal Analysis" by I. Moerdijk and G. Reyes, Springer-Verlag 1991); our result may be seen as an alternative and much simpler approach to these objects.

Finally, we give a short overview over problems and further topics in the context of general Lie theory. In general, it is not possible to integrate differential equations in our general context (this is known to be so already in the p-adic case or in the locally convex real case), and so most problems take the form of "integration problems". For instance, for a general Lie group over k, there is no exponential map, but pushing the theory of connections somewhat further than usual one can define a certain bundle isomorphism on the level of higher order tangent bundles which serves to replace the missing exponential map (work in progress, cf. [3]). Then one may ask whether there is also an analog of the Campbell-Hausdorff formula. This seems to be indeed the case, but the precise form of this formula is unknown at present (note that the characteristic of k is arbitrary). The ultimate integration problem in Lie theory would be to find an analog of "Lie's third theorem" in our general context, i.e. to find necessary and sufficient conditions for a Lie algebra to be "integrable" to a Lie group. This problem can also be posed for symmetric spaces and Lie triple systems. Remarkably enough, for *Jordan algebraic structures* the integration problem can be solved (cf. [2], [4]): there is a functor assigning to every Jordan-structure over k (-algebra, -triple system or -pair, satisfying some natural continuity condition) a geometry which is smooth over k. This is possible since "Jordan geometries" tend to be algebraic, whereas "Lie geometries" only tend to be analytic.

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Cohomology of holomorphic vector fields on a punctured Riemann surface

Friedrich Wagemann

Let Σ denote a compact Riemann surface of genur g and $\Sigma_r = \Sigma \setminus \{p_1, \ldots, p_r\}$ a punctured Riemann surface, punctured in $r \geq 1$ distinct points.
Let $Hol(\Sigma_r)$ denote the Lie algebra of holomorphic vector fields on Σ_r. It is a topological Lie algebra with respect to the topology of uniform convergence on compact sets in Σ_r. The underlying topological space is Fréchet.
The goal of this survey is an Ext-description of the continuous cohomology of $Hol(\Sigma_r)$, i.e. to describe it in terms of (topologically split) exact sequences of $Hol(\Sigma_r)$-modules.
In a first section, we recalled the setting of continuous cohomology of a Fréchet Lie algebra $\mathfrak{g}$ [1]. The Ext-description, which is standard for ordinary cohomology by work of Yoneda, is more difficult here as there is no standard category of topological $\mathfrak{g}$-modules which posesses enough projectives and injectives. Nevertheless, $H^2(\mathfrak{g}, \mathbb{C})$ classifies central extensions which are topologically split (i.e. split as sequences of topological vector spaces). Our first theorem [6] is that the standard map associating to a (topologically split) crossed module its continuous 3-cocycle induces a bijection of the set of crossed modules of $\mathfrak{g}$ with V to $H^3(\mathfrak{g}, V)$ in case there is a topologically split exact sequence $0 \to V \to W \to U \to 0$ such that $H^3(\mathfrak{g}, W) = 0$.
In a second section, we recalled N. Kawazumi's theorem [2] on the continuous cohomology of $Hol(\Sigma_r)$. It states that $H^*(Hol(\Sigma_r), \mathbb{C})$ is isomorphic to the singular cohomology of the space $Map(\Sigma_r, S^3)$ of continuous maps from Σ_r to the 3-sphere S^3, equipped as a topological space with the compact-open topology. The latter cohomology algebra is a graded commutative Hopf algebra in N generators of degree 2 (where N equals the dimension of $H^1(\Sigma_r)$) and one generator of degree 3, a kind of Godbillon-Vey generator. We generalized Kawazumi's work to n dimensional complex manifolds [4], and showed also that one can obtain from it the continuous cohomology of the topological subspace of meromorphic vector fields [3] (i.e. those holomorphic vector fields on Σ_r which have at most poles in $p_1, \ldots, p_r$) which play an important rôle in Krichever-Novikov's approch to string theory.
In a third section, we showed in our main theorem how to construct a crossed module representing the mentioned Godbillon-Vey type generator [5]. The corresponding 4-term exact sequence is constructed by splicing together the short exact de Rham sequence of holomorphic differential forms on the universal cover $\widetilde{\Sigma_r}$ of Σ_r, say

$$0 \to \mathbb{C} \to \Omega^0(\widetilde{\Sigma_r}) \to \Omega^1(\widetilde{\Sigma_r}) \to 0,$$

and an abelian extension of $Hol(\Sigma_r)$ by $\Omega^1(\widetilde{\Sigma_r})$ by a certain 2-cocycle.

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On the holomorphic structure of G-orbits in compact hermitian symmetric spaces

Wilhelm Kaup

In this lecture we give a survey on the results of the submitted paper [4]. Let us start with a complex Banach space E of dimension n (that is $\mathbb{C}^n$ with a fixed norm $\| \, \|$). The open unit ball $D \subset E$ is called a *bounded symmetric domain* if the group $G := \mathrm{Aut}(D)$ of all biholomorphic automorphisms of D acts transitively on D (this is not an essential restriction to the usual more abstract definition, see e.g. [2]). Then it is well known that G is a semi-simple Lie group and that the isotropy subgroup $K\colon = \{g \in G : g(0) = 0\}$ is a maximal compact subgroup coinciding with the group of all linear isometries of the complex Banach space E. The compact dual Z of D in the sense of symmetric hermitian spaces is a compact homogeneous complex manifold containing E as open dense subset in such a way that $G \cong \{g \in \mathrm{Aut}(Z) : g(D) = D\}$ (Z is the Riemann sphere in case $E = \mathbb{C}$ and D the open unit disk). In this sense G also acts on Z by biholomorphic transformations and has only finitely many orbits there (one of which is the domain $D \subset Z$, another one is the Shilov boundary of D, the unique closed G-orbit in Z).

The G-orbits in Z as homogeneous spaces and the holomorphic arc components of their closures have been described explicitly in [5]. Here we are interested in the Cauchy-Riemann structure of G-orbits (which for open orbits is just the usual holomorphic structure as complex manifold). For fixed orbit $M\colon = G(a)$, $a \in Z$, let us briefly recapitulate its CR-structure (take [1] as general reference for arbitrary CR-manifolds): For every $c \in M$ the tangent space T_cM is canonically contained in the tangent space T_cZ, which is a complex vector space in a natural way. Clearly, $H_cM\colon = T_cM \cap i\,T_cM$ (called the *holomorphic tangent space at* $c \in M$) is the biggest complex subspace contained in T_cM. The CR-structure on M is given by the complex vector bundle $HM \subset TM$. In particular, a smooth function $f\colon M \to \mathbb{C}$ (or more generally with values in another CR-manifold) is called CR if it satisfies the tangential Cauchy-Riemann partial differential equations in the sense that the differential $df(c)\colon T_cM \to \mathbb{C}$ is complex linear on every holomorphic tangent space H_cM, $c \in M$. Here we are interested in the holomorphic extendibility of CR-functions, the explicit determination of CR-automorphism groups and the CR-equivalence problem for G-orbits.

For simplicity and without essential loss of generality we restrict to the case where the bounded symmetric domain D is *irreducible*, that is, not a direct product of lower dimensional bounded symmetric domains. Then, if D has rank r, there exist precisely $\binom{r+2}{2}$ G-orbits in Z, which can be indexed in a canonical way as $M_{p,q}$ with integers $p, q \geq 0$ satisfying $p + q \leq r$ (compare the special example below). There are precisely $r+1$ open orbits (those with $p+q = r$) and also $r+1$ orbits (those with $q = 0$) contained in the closure $\overline{D}$ of D. In case D is of tube type, the Shilov boundary $M_{0,0}$ of D is totally real in Z, and there is a biholomorphic transformation ι of Z with period 2, mapping every $M_{p,q}$ onto $M_{q,p}$, thus giving a real-analytic CR-equivalence between $M_{p,q}$ and $M_{q,p}$. It's the existence of this transformation ι that is responsible for some extra phenomena in the tube case. For a presentation of our results therefore assume in the following that the irreducible bounded symmetric domain D is not of tube type: Then, if $M = M_{p,0}$ (that is, $M \subset \overline{D}$), every continuous CR-function f on M has a unique continuous extension $\hat{f}$ to the linear convex hull $\hat{M} = \bigcup_{k \geq p} M_{k,0}$ of M that is holomorphic on the domain $D = M_{r,0}$, and $\hat{M}$ is maximal in Z with respect to this extension property. For every other orbit $M = M_{p,q}$, $q > 0$, every continuous CR-function on M is constant and every continuous CR-function on $M \cap E$ has a unique holomorphic extension to E, implying that then every infinitesimal CR-transformation of M extends to a holomorphic vector field on Z. This can be used to show for every G-orbit M in Z that the group $\mathrm{Aut}(M)$ of all CR-automorphisms of M is just the group G and also that the G-orbits in Z are pairwise CR-inequivalent. The proofs use extensively the Jordan algebraic description of bounded symmetric domains as well as the CR-extension results for K-orbits obtained in [3].

For the announced example fix integers $s \geq r \geq 1$ in the following and denote by $E\colon = \mathbb{C}^{s\times r}$ the Banach space of all complex $s \times r$–matrices, where $\|z\|$ is the operator norm of the matrix z, considered as a linear operator $\mathbb{C}^r \to \mathbb{C}^s$. Then the open unit ball $D \subset E$ is an irreducible bounded symmetric domain of rank r, and D is of tube type if and only if $s = r$. The subgroup $\mathrm{SU}(s,r) \subset \mathrm{SL}(s+r,\mathbb{C})$ acts by linear fractional transformations transitively on D in the following way: Write every $g \in \mathrm{SU}(s,r)$ in block form $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $a \in \mathbb{C}^{s\times s}$, $b \in \mathbb{C}^{s\times r}$, $c \in \mathbb{C}^{r\times s}$, $d \in \mathbb{C}^{r\times r}$ and put $g(z) := (az+b)(cz+d)^{-1}$ for all $z \in D$. Then the connected identity component of $G = \mathrm{Aut}(D)$ consists of all transformations obtained this way. The compact dual Z of D is the Grassmann manifold $\mathbb{G}_{r,s}$ of all r-planes in $\mathbb{C}^r \times \mathbb{C}^s$, in which E is embedded by identifying every matrix $z \in E$ with its graph $\{(x, zx) : x \in \mathbb{C}^r\} \in \mathbb{G}_{r,s}$. For every $z = (z_{jk}) \in E$ let $z^*\colon = (\overline{z_{jk}}) \in \mathbb{C}^{r\times s}$ and $\mathbb{1}$ the unit matrix in $\mathbb{C}^{r\times r}$. Then, if the hermitian matrix $\mathbb{1} - z^*z \in \mathbb{C}^{r\times r}$ has type (p,q) (meaning p positive and q negative eigenvalues), we have $G(z) = M_{p,q}$. In particular,

$$D = M_{r,0} = \{z \in E : \mathbb{1} - z^*z \text{ positive definite}\}$$

and

$$M_{0,0} = \{z \in E : z^*z = \mathbb{1}\}$$

is the Shilov boundary of D. In the tube case, i.e. $r = s$, the involutory transformation ι leaves $\mathrm{GL}(r,\mathbb{C}) \subset Z$ invariant and satisfies $\iota(z) = z^{-1}$ there.

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Deformation quantization of Kähler manifolds

Martin Schlichenmaier

In this talk I presented results on the Berezin-Toeplitz deformation quantization for compact quantizable Kähler manifolds. Some of them were obtained jointly with M. Bordemann and E. Meinrenken. Some of them jointly with A. Karabegov.

Let (M, ω) be a Kähler manifold and $(C^\infty(M), \cdot)$ the associative and commutative algebra of C^∞-functions under the pointwise product. This algebra is endowed with a Poisson structure via $\{f, g\} := \omega(H_f, H_g)$, with H_f the Hamiltonian vector field defined by $\omega(H_f, .) = df(.)$. A formal deformation quantization or a star product is an associative product $\star$ on the vector space of formal power series $C^\infty(M)[[\nu]]$, which is ν-adically continuous and fulfills

$$(1) \quad f \star g = f \cdot g \bmod \nu, \qquad (2) \quad \frac{1}{\nu}(f \star g - g \star f) = \mathrm{i}\{f, g\} \bmod \nu.$$

In particular,

$$f \star g = \sum_{j=0}^{\infty} \nu^j C_j(f, g),$$

with bilinear maps $C_j : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$. A star product is called a differential star product if the C_j are bidifferential operators. Usually one assumes also $f \star 1 = 1 \star f = f$. Two star products $\star$ and $\star'$ (for the same Poisson structure) are called equivalent if there is an isomorphism of algebras B (i.e. $B(f) \star' B(g) = B(f \star g)$) such that the formal sum $B = \sum_{j=0}^{\infty} \nu^j B_j$ starts with $B_0 = id$. A differential star product is called a star product with the property of "separation of variables" (in the terminology of Karabegov) or of Wick-type (in the terminology of Bordemann-Waldmann) if in the first argument of C_j only holomorphic and in the second argument only anti-holomorphic derivatives appear. In joint work with A. Karabegov I showed that the Berezin-Toeplitz (BT) deformation quantization is a differential star product with the separation of variables property [KS].

The approach presented here works for arbitrary compact and quantizable Kählermanifolds. A Kähler manifold is called quantizable if there exists a holomorphic hermitian line bundle over M: (L, h, ∇), (∇ is the compatible connection) such that $curv_{L,\nabla} = -\mathrm{i}\omega$. Important examples of such quantizable Kähler manifolds are given by the projective space with the hyperplane section bundle, projective submanifolds, abelian varieties, moduli spaces of flat $su(N)$-connections on a Riemann surface, moduli spaces of stable algebraic vector bundles of rank N and degree d over an algebraic curve, etc.

The metric h on L extends to $h^{(m)}$ on $L^m := L^{\otimes m}$. By integrating it against the Liouville form it defines a scalar product on the space of C^∞ sections. Inside the L^2 completion there is the finite-dimensional subspace $\Gamma^{(m)}_{hol}$ of holomorphic sections. Let $\Pi^{(m)}$ be the projection onto this subspace. The BT quantum operators associated to a function f on M are defined as $(T^{(m)}_f)_{m\in\mathbb{N}}$ with

$$T^{(m)}_f : \Gamma^{(m)}_{hol} \to \Gamma^{(m)}_{hol}; \quad s \mapsto T^{(m)}_f(s) = \Pi^{(m)}(f \cdot s)\ .$$

Theorem 1. [BMS].
(a) $\lim_{m\to\infty} ||T^{(m)}_f|| = ||f||_{sup}$.
(b) $||m\mathrm{i}[T^{(m)}_f, T^{(m)}_g] - T^{(m)}_{\{f,g\}}|| = O(1/m)$,
(c) $||T^{(m)}_f \cdot T^{(m)}_g - T^{(m)}_{f\cdot g}|| = O(1/m)$.

Theorem 2. [Bia], [BMS], [CMF]. *There exists a unique star product* $f \star_{BT} g = \sum_{k=0}^\infty \nu^k C_k(f,g)$, *such that*

$$T^{(m)}_f \cdot T^{(m)}_g \sim \sum_{k=0}^{\infty} \left(\frac{1}{m}\right)^k T^{(m)}_{C_k(f,g)}, \quad m \to \infty.$$

This star product is called the Berezin-Toeplitz star product.

Theorem 3. [KS]. *The BT-star product is a differential star product with the separation of variables property. It has as Karabegov classifying form* $\tilde{\omega}_{BT} = -\frac{1}{\lambda}\omega + \omega_{can}$ *and as Fedosov-Deligne class* $c(\star_{BT}) = \frac{1}{\mathrm{i}}\left(\frac{1}{\lambda}[\omega] - \frac{\epsilon}{2}\right)$.

Here λ is a formal variable, ω_{can} is the curvature form of the canonical holomorphic line bundle and ϵ is the canonical class. Furthermore, it should be recalled that all star products with the separation of variables property are uniquely given by their (formal) Karabegov form, and all differential star products up to equivalence given by their (formal) Fedosov-Deligne class. As an important tool in the proof of the last theorem the Berezin transform $I^{(m)} : C^\infty(M) \to C^\infty(M)$ was used. With the help of the (suitably generalized) Berezin covariant symbol map σ it can be described as $I^{(m)}(f) = \sigma(T^{(m)}_f)$. In [KS] it was shown that it has a complete asymptotic expansion in $1/m$ which starts with $f(x) + (1/m)\Delta f + \ldots$

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The generalized Cayley map from an algebraic group to its Lie algebra
Peter W. Michor

This talk is mainly based on the paper [4].

Let $\pi : G \to \mathrm{End}(V)$ be an infinitesimally faithful complex representation of a connected Lie group G. Consider $(A, B) \mapsto \mathrm{tr}(AB)$ on $\mathrm{End}(V)$ and suppose that it is non-degenerate on the linear subspace $\pi'(\mathfrak{g}) \subseteq \mathrm{End}(V)$. Then the orthogonal projection $\mathrm{pr}_\pi : \mathrm{End}(V) \to \pi'(\mathfrak{g})$ is defined:

$$\begin{tikzcd} G \arrow[r, "\text{representation } \pi"] \arrow[d, "\Phi_\pi \text{ Caley map}"'] & \mathrm{End}(V) \arrow[d, "\mathrm{pr}_\pi \text{ orthoproj.}"] \\ \mathfrak{g} \arrow[r, tail, "\text{infinites. repr. } \pi'"'] & \pi'(\mathfrak{g}) \end{tikzcd} \qquad \Psi_\pi(g) = \Psi(g) := \det(d\Phi(g))$$

The Cayley mapping Φ has the following simple properties:

(1) $\Phi(bxb^{-1}) = \mathrm{Ad}_b(\Phi(x))$.
(2) We have $\Phi(g) \in \mathrm{Cent}(\mathfrak{g}^g) \subset Z_{\mathfrak{g}}(\mathfrak{g}^g)$.
(3) $d\Phi(e) : \mathfrak{g} \to \mathfrak{g}$ is the identity mapping.
(4) $H \subset G$ be a Cartan subgroup with Cartan algebra $\mathfrak{h} \subset \mathfrak{g}$. Then $\Phi(H) \subset \mathfrak{h}$.
(5) For the character $\chi_\pi(g) = \mathrm{tr}(\pi(g))$ of π we have $d\chi_\pi(g)(T_e(\mu_g)X) = \mathrm{tr}(\pi'(\Phi_\pi(g))\pi'(X))$

Further results are:

• Let $\pi : G \to \mathrm{Aut}(V)$ be a representation admitting a Cayley mapping. Let $H = (\bigcap_{a \in A} G^a)_o = (G^A)_o \subseteq G$ be a subgroup which is the connected centralizer of a subset $A \subseteq G$ and suppose that H is itself reductive. Then $\pi|H : H \to \mathrm{End}(V)$ admits a Cayley mapping and $\Phi_\pi|H = \Phi_{\pi|H} : H \to \mathfrak{h}$.

• Let G be a semisimple real or complex Lie group, let $\pi : G \to \mathrm{Aut}(V)$ be an infinitesimally effective representation. Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$ be the decomposition into the simple ideals $\mathfrak{g}_i$. Let $G_1, \ldots, G_k$ be the corresponding connected subgroups of G. Then $\Phi_\pi|G_i = \Phi_{\pi|G_i} \quad$ for $i = 1, \ldots, k$.

• G a simple Lie group, for direct sum and tensor product representations

$$\Phi_{\pi_1 \oplus \pi_2}(g) = \frac{j_{\pi_1}}{j_{\pi_1 \oplus \pi_2}} \Phi_{\pi_1}(g) + \frac{j_{\pi_2}}{j_{\pi_1 \oplus \pi_2}} \Phi_{\pi_2}(g) \in \mathfrak{g}.$$

$$\Phi_{\pi_1 \otimes \pi_2}(g) = \frac{j_{\pi_1} \chi_{\pi_2}(g)}{j_{\pi_1 \otimes \pi_2}} \Phi_{\pi_1}(g) + \frac{\chi_{\pi_1}(g) j_{\pi_2}}{j_{\pi_1 \otimes \pi_2}} \Phi_{\pi_2}(g) \in \mathfrak{g}.$$

Results for algebraic groups. Now let G be a reductive complex algebraic group and π a rational representation. We have $A(\mathfrak{g}) = A(\mathfrak{g})^G \otimes \mathrm{Harm}(\mathfrak{g})$, where $\mathrm{Harm}(\mathfrak{g})$ is the space of all regular functions killed by all invariant differential operators with constant coefficients. We define $\mathrm{Harm}_\pi(G) := \Phi_\pi^*(\mathrm{Harm}(\mathfrak{g}))$. It is a G-module.

• For the localization at Ψ we have $A(G)_\Psi = A(G)^G_\Psi \otimes \mathrm{Harm}_\pi(G)$. Moreover, we have $A(G) = A(G)^G \otimes \mathrm{Harm}_\pi(G)$ if and only if $\Phi : G \to \mathfrak{g}$ maps regular orbits in G to regular orbits in $\mathfrak{g}$.
• If $\Phi(e) = 0 \in \mathfrak{g}$ then for the G-equivariant extension of the rational function fields $\Phi^* : Q(\mathfrak{g}) \to Q(G)$ the degrees satisfy $[Q(G) : Q(\mathfrak{g})] = [Q(G)^G : Q(\mathfrak{g})^G]$.
• Let $a \in G$ be regular. Assume that $d\Phi(a)$ is invertible. Then Φ restricts to an isomorphism $\Phi : \overline{\mathrm{Conj}_G(a)} \to \overline{\mathrm{Ad}_G(\Phi(a))}$ of affine varieties.
• Let $a \in G$. Then for the semisimple parts we have $\Phi(a_s) = \Phi(a)_s$ and $\Phi(a) = \Phi(a_s) + \Phi(a)_n \in \mathfrak{g}^a$ is the Jordan decomposition.
• Let G be a connected reductive complex algebraic group and let $\Phi : G \to \mathfrak{g}$ be the Cayley mapping of a rational representation with $\Phi(e) = 0$. Then $\Phi : G_{\mathrm{pos}} \to \mathfrak{g}_{\mathrm{real}}$ is bijective and a fiber respecting isomorphism of real algebraic varieties, where G_{pos} is the set of all $a \in G$ whose semisimple part has positive eigenvalues, and $\mathfrak{g}_{\mathrm{real}}$ is the set of all $X \in \mathfrak{g}$ whose semisimple part has only real eigenvalues.

Relation to the classical Cayley mapping. Let $T : \mathrm{Spin}(n, \mathbb{C}) \to SO(n, \mathbb{C})$ be the double cover. We consider the spin representation $\mathrm{Spin} : \mathrm{Spin}(n, \mathbb{C}) \to \mathrm{Aut}(S_n)$.
• There is a choice of the sign of the square root so that $\chi(g) := \sqrt{\det(1 + T(g))}$ satisfies
$$\Phi_{\mathrm{Spin}}(g) = -\frac{2}{2^{n/2}}\,\chi(g)\,\Gamma(T(g)) \in \mathfrak{so}(n, \mathbb{C}).$$
for all $g \in \mathrm{Spin}(n, \mathbb{C})$. Moreover, $\chi \in A(\mathrm{Spin}(n, \mathbb{C}))$ and we have for the rational function fields
$$Q(\mathrm{Spin}(n))^{\mathrm{Spin}(n)} = Q(\mathfrak{so}(n, \mathbb{C}))^{\mathrm{Spin}(n)}[\chi],$$
$$Q(\mathrm{Spin}(n)) = Q(\mathfrak{so}(n, \mathbb{C}))[\chi].$$
Thus the generalized Cayley mapping $\Phi_{\mathrm{Spin}} : \mathrm{Spin}(n, \mathbb{C}) \to \mathfrak{so}(n, \mathbb{C})$ factors to the classical Cayley transform $\Gamma : SO(n, \mathbb{C})^* \to \mathrm{Lie}\,\mathrm{Spin}(n, \mathbb{C})^{(*)}$, up to multiplication by a function, via the natural identifications.

Relation to Poisson structures. For a representation π of a Lie group G we can try to pull back the Poisson structure on $\mathfrak{g}^*$ via the derivative of the character $d\chi_\pi : G \to \mathfrak{g}^*$. This pullback is a rational Poisson structure on G which in fact is an integrable Dirac structure in the sense of [1], [2], [3]. Let us explain this a little:

Let M be a smooth manifold of dimension m. A *Dirac structure* on M is a vector subbundle $D \subset TM \times_M T^*M$ with the following two properties:

(1) Each fiber D_x is maximally isotropic with respect to the metric of signature (m, m) on $TM \times_M T^*M$ given by $\langle (X, \alpha), (X', \alpha') \rangle_+ = \alpha(X') + \alpha'(X)$. So D is of fiber dimension m.
(2) The space of sections of D is closed under the non-skew-symmetric version of the Courant-bracket $[(X, \alpha), (X', \alpha')] = ([X, X'], \mathcal{L}_X \alpha' - i_{X'} d\alpha)$.

Natural examples of Dirac structures are the following: Symplectic structures ω on M, where $D = D^\omega = \{(X, \omega(X)) : X \in TM\}$ is just the graph of $\omega : TM \to T^*M$; these are precisely the Dirac structures D with $TM \cap D = \{0\}$. Poisson structures

P on M where $D = D^P = \{(P(\alpha), \alpha) : \alpha \in T^*M\}$ is the graph of $P : T^*M \to TM$; these are precisely the Dirac structures D which are transversal to T^*M.

Given a Dirac structure D on M we consider its range $R(D) = \mathrm{pr}_{TM}(D) = \{X \in TM : (X, \alpha) \in D \text{ for some } \alpha \in T^*M\}$. There is a skew symmetric 2-form Θ_D on $R(D)$ which is given by $\Theta_D(X, X') = \alpha(X')$ where $\alpha \in T^*M$ is such that $(X, \alpha) \in D$. The range $R(D)$ is an integrable distribution of non-constant rank in the sense of Stefan and Sussmann, see [5], so M is foliated into maximal integral submanifolds L of $R(D)$ of varying dimension, which are all initial submanifolds. The form Θ_D induces a closed 2-form on each leaf L and (L, Θ_D) is thus a presymplectic manifold (Θ_D might be degenerate on L). If the Dirac structure corresponds to a Poisson structure then the (L, Θ_D) are exactly the symplectic leaves of the Poisson structure.

The main advantage of Dirac structures is that one can apply arbitrary push forwards and pull backs to them. So if $f : N \to M$ is a smooth mapping and D_M is a Dirac structure on M then the pull back is defined by $f^*D_M = \{(X, f^*\alpha) \in TN \times_N T^*N : (Tf.X, \alpha) \in D_M\}$. Likewise the push forward of a Dirac structure D_N on N is given by $f_*D_N = \{(Tf.X, \alpha) \in TM \times_M T^*M : (X, f^*\alpha) \in D_N\}$.

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Θ-hypergeometric functions and shift operators

Angela Pasquale

The noncompactly causal (NCC) symmetric spaces are a small but nice class of pseudo-Riemannian symmetric spaces. The interest in these spaces was raised by the studies on the global structure of the space-time (see for instance [5]). In 1994, Faraut, Hilgert and Olafsson [1] could exploit the geometry of these spaces to extend to them the theory of spherical functions, which Harish-Chandra had developed in the late 50s on the Riemannian symmetric spaces of noncompact type [4]. As in the Riemannian case, the spherical functions on a NCC symmetric space G/H are the (suitably normalized) smooth H-invariant joint eigenfunctions of the commutative algebra of G-invariant differential operators on G/H. However, due to the non-compactness of H, they turn out to be much less regular than those of Harish-Chandra: they are only defined on an open submanifold of G/H; they are meromorphic (not entire) in the spectral parameter; they can be described by integral formulas only for certain values of the spectral parameters. Many of the difficulties encountered when studying the spherical functions on NNC symmetric spaces can be overcome by working in the more general setting of Θ-spherical functions.

The Θ-hypergeometric functions are special functions associated with root systems that generalize the spherical functions on both the NCC and the Riemannian symmetric spaces. Their definition has been suggested by Olafsson's expansion formula [7] for the spherical functions on a NCC symmetric space G/H. This formula shows that the restriction of the spherical functions of G/H to a specific Weyl chamber of Cartan subgroup is a certain linear combination of Harish-Chandra series for the Riemannian dual G/K. In their theory of hypergeometric functions associated with root systems [3, 2, 10, 6, 11], Heckman and Opdam developed very powerful methods for studying this kind of linear combinations without relying on a Riemannian structure. It is then quite natural to to try to extend Heckman-Opdam's definitions and methods to enclose also the spherical functions on NCC symmetric spaces. The big family of special functions originating from this extension gives precisely the Θ-hypergeometric functions. They are constructed from a triple $(\mathfrak{a}, \Sigma, m)$, where $\mathfrak{a}$ is a Euclidean symmetric space, Σ is a root system in the dual $\mathfrak{a}^*$ of $\mathfrak{a}$, and m is a multiplicity functions on Σ. As the hypergeometric functions associated with root systems, the Θ-hypergeometric functions are joint eigenfunctions of the hypergeometric system of Heckman and Opdam. The parameter Θ designates a subset of a fixed fundamental system Π of positive simple roots in Σ. The different choices of Θ lead to a lattice of special functions associated with the given root system. At the top of the lattice, that is for $\Theta = \Pi$, we find the hypergeometric functions of Heckman and Opdam; at the bottom, that is for $\Theta = \emptyset$, (certain multiples of) the Harish-Chandra series. In the middle appear many new special functions. For "geometric" triples $(\mathfrak{a}, \Sigma, m)$, the Θ-hypergeometric functions corresponding to $\Theta = \Pi$ yield Harish-Chandra's spherical functions, whereas the spherical functions on NCC symmetric spaces arise from some of the new special functions in the central part of the lattice. This unified framework allows us, for instance, to derive information on the spherical functions on NCC symmetric spaces from those of the spherical functions of the Riemannian dual.

A particularly nice situation occurs for even multiplicity functions on reduced root systems. Geometrically, this situation corresponds to Riemannian symmetric spaces G/K with the property that all Cartan subalgebras in the Lie algebra $\mathfrak{g}$ of G are conjugate. The simplest example is when $\mathfrak{g}$ admits a complex structure, in which case all multiplicities are equal to 2. The analysis of Θ-hypergeometric functions with even multiplicities is simplified by the use of Opdam's shift operators (see e.g. [6]). By modifying one of these operators, it is possible to obtain a Weyl-group-invariant differential operator with regular coefficients yielding Θ-hypergeometric functions with even multiplicities from averages of exponential functions. In particular, this provides new formulas for the spherical functions on

Riemannian symmetric spaces with even multiplicities of both noncompact and compact type. The study of of the Θ-hypergeometric functions in even multiplicities and their associated harmonic analysis is a joint work with Gestur Ólafsson [9].

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Maximal adapted complexifications of Riemannian homogeneous spaces

Andrea Iannuzzi
(joint work with Stefan Halverscheid)

For a Riemannian real-analytic manifold M one can construct canonical complexifications by defining the *adapted* complex structure on a domain of the tangent bundle TM, as shown by Guillemin-Stenzel and Lempert-Szoeke ([GS], [LS]). This uniquely determines the complexification in a neighborhood of M, which is identified with the zero section in TM, however in general there are questions about existence and unicity of a maximal domain Ω_{max} on which the adapted complex structure exists. If Ω_{max} is understood, by functoriality of the definition it may be regarded as an invariant of the metric, i.e., isometric manifolds have biholomorphic maximal domains. For instance examples are given by symmetric spaces of non-compact type ([BHH]), compact normal Riemannian Homogeneous spaces ([Sz2]), compact symmetric spaces ([Sz1]) and spaces obtained by Kählerian reduction of these ([A]). Note that in the mentioned cases maximal domains turn out to be Stein.

Let us consider a Riemannian homogeneous space $M = G/K$, with G a Lie group of isometries and K compact. It is reasonable to assume that $\dim_{\mathbb{C}} G^{\mathbb{C}} = \dim_{\mathbb{R}} G$, where $G^{\mathbb{C}}$ is the universal complexification of G. Then $K^{\mathbb{C}}$ acts on $G^{\mathbb{C}}$, the left action on M induces a natural G-action on TM and under certain extensibility assumptions on the geodesic flow of M one obtains a real-analytic and G-equivariant map $P : TM \to G^{\mathbb{C}}/K^{\mathbb{C}}$ such that

the connected component of the non-singular locus of DP containing M is the unique maximal domain on which the adapted complex structure exists.

This applies to the case of generalized Heisenberg groups and naturally reductive Riemannian homogeneous spaces, among which one finds all isotropy irreducible homogeneous spaces classified by J. Wolf [W].

As an application it is shown that for all generalized Heisenberg groups such maximal domain is neither holomorphically separable, nor holomorphically convex. We are not aware of previous non-Stein examples. Moreover allready in the case of the 3-dimensional Heisenberg group one notices mixed signature Ricci curvature, suggesting an influence of curvature properties of M on the complex geometry of the maximal adapted complexification. Some recent new examples give a different light to such point of view.

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