Mathematisches Forschungsinstitut Oberwolfach

Report No. 8/2004

Mini-Workshop: Nonlinear Spectral and Eigenvalue Theory with Applications to the p-Laplace Operator

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February 15th – February 21st, 2004

Introduction by the Organisers

What is the state-of-the-art of abstract spectral and eigenvalue theory for nonlinear operators, and how may this theory be applied to nonlinear equations involving the p-Laplace operator? These two questions have provided the main focus of the Mini-Workshop. Accordingly, the main topics covered by the talks on this Mini-Workshop have been

- spectra for nonlinear operators,
- nonlinear eigenvalue problems, and
- equations involving the p-Laplace operator.

Of course, these three topics are not mutually independent, but there are various interconnections between them which are of particular interest. For example, sets of eigenvalues (point spectra) may be regarded, as in the linear case, as an important part of the spectrum; conversely, nonlinear eigenvalue theory is one of the historical roots of nonlinear spectral theory. Moreover, the p-Laplace operator is one of the most interesting homogeneous (though nonlinear) operators which may not only serve as a "model operator" in nonlinear eigenvalue problems, but also occurs quite frequently in various applications to physics, mechanics, and elasticity.

The aim and scope of the Mini-Workshop was to bring together experts on nonlinear spectral analysis and operator theory, on the one hand, and more application-oriented specialists in eigenvalue problems for nonlinear partial differential equations (like the p-Laplace equation), on the other. As a result, 15 leading experts in the field from 10 different countries discussed recent progress and open problems in the theory, methods, and applications of spectra and eigenvalues of nonlinear operators.

Mini-Workshop: Nonlinear Spectral and Eigenvalue Theory with Applications to the p-Laplace Operator

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Abstracts

Asymmetric Eigenvalue Problems with Weights for the p-laplacian with Neumann Boundary Conditions

M. Cuesta (Calais)
(joint work with M. Arias (Granada), J.-P. Gossez (Bruxelles))

The motivation of this work is the study of

$$(1) \qquad -\Delta_p u = f(x,u) \text{ in } \Omega, \quad \tfrac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega,$$

where $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $1 < p < \infty$, and Ω is a bounded smooth domain of $\mathbb{R}^N$ and $|f(x,s)| \le a(x)|s|^{p-1} + b(x)$ with a, b belonging to suitable Lebesgue spaces. Our ultimate goal is to find optimal conditions on the limits at $+\infty$ and $-\infty$ of the quotients $f(x,s)/|s|^{p-2}s$ and $pF(x,s)/|s|^p$ (where $F(x,s) := \int_0^s f(x,t)\,dt$) as $s \to +\infty$ and $s \to -\infty$ to assure solvability of (1). When considering $m(x) = \lim_{s\to+\infty} \frac{f(x,s)}{|s|^{p-2}s}$, $n(x) = \lim_{s\to-\infty} \frac{f(x,s)}{|s|^{p-2}s}$, we are lead to study *weighted asymmetric eigenvalue problems* of the form

$$(2) \qquad -\Delta_p u = \lambda(m(x)(u^+)^{p-1} - n(x)\,(u^-)^{p-1}) \text{ in } \Omega, \quad \tfrac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega$$

We will always assume that the weights $m(x)$ and $n(x)$ are possibly non constant, different, indefinite and belong to $L^r(\Omega)$ where $r > N/p$ if $p \le N$ and $r = 1$ if $p > N$. We will also assume that m^+ and $n^+ \not\equiv 0$ and we are only interested on *positive eigenvalues*. Notice that 0 is always an eigenvalue of (2).

The case $m(x) \equiv n(x)$ have been studied [5]. When $m(x)$ are $n(x)$ are constant and different, (2) leads to the notion of Fučik spectrum and the so-called problems of Ambrosetti-Prodi type. Analogous problems (1) and (2) have been treated with *Dirichlet boundary conditions* by [1].

The study of (2) start with the following symmetric eigenvalue problem

$$(3) \qquad -\Delta_p u = \lambda m(x)|u|^{p-2}u \text{ in } \Omega, \quad \tfrac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega.$$

The following value introduced by [5] plays a crucial role:

$$\lambda^*(m) := \inf\{\int_\Omega |\nabla u|^p \; dx \; : \; \int_\Omega m\,|u|^p \; dx = 1\}.$$

which satisfies: (1) If $\int_\Omega m\,dx < 0$ then $\lambda^*(m) > 0$ is the unique non zero principal eigenvalue, it admits a non negative eigenfunction and there is no eigenvalue on $]0, \lambda^*(m)[$. (2) If $\int_\Omega m\,dx > 0$ then $\lambda^*(m) = 0$ is the unique non negative principal eigenvalue and (3) If $\int_\Omega m\,dx = 0$ then $\lambda^*(m) = 0$ is the unique principal eigenvalue. Besides a sequence of eigenvalues can be constructed using the Ljusternik-Schnirelmann critical point theory, cf. [3].

It follows straightforward that the principal eigenvalues of (2) are $\lambda = \lambda_1(m)$ and $\lambda = \lambda_1(n)$. We present in this work a construction of a *non principal eigenvalue of* (2) by considering the functionals $A(u) := \int_\Omega |\nabla u|^p$, $B_{m,n}(u) : \int_\Omega (m(u^+)^p +$

$n(u^-)^p)$ and $\tilde{A}$ the restriction of A to the C^1 manifold $M_{m,n} := \{u \in W_0^{1,p}(\Omega) : B_{m,n}(u) = 1\}$. We prove

Theorem 1. *Let* $\Gamma := \{\gamma \in C([0,1], M_{m,n}) : \gamma(0) \leq 0$ *and* $\gamma(1) \geq 0\}$. *Then*

(1) $\Gamma \neq \emptyset$.

(2) *The value* $c(m,n) := \inf_{\gamma\in\Gamma} \max_{u\in\gamma[0,1]} \tilde{A}(u)$ *is an eigenvalue of* (2) *which satisfies*

$$c(m,n) > \max\{\lambda^*(m), \lambda^*(n)\}.$$

(3) *There is no eigenvalues of* (2) *between* $\max\{\lambda^*(m), \lambda^*(n)\}$ *and* $c(m,n)$.

The proof of this theorem relies on a critical point theorem of [2] for C^1 functionals restricted to C^1-manifolds that satisfy the *Palais-Smale condition of Cerami* (denoted (PSC) for short). This is one of main issues of the paper. Presicely we can prove that (1) $\tilde{A}$ satisfies $(PS)_c$ along bounded sequences $\forall c \geq 0$, (2) $\tilde{A}$ satisfies $(PSC)_c$ $\forall c > 0$, (3) if $\int_\Omega m\,dx \neq 0$ and $\int_\Omega n\,dx \neq 0$ then $\tilde{A}$ satisfies $(PS)_c$ for all $c \geq 0$, (4) if $\int_\Omega m\,dx = 0$ or $\int_\Omega n\,dx = 0$ then $\tilde{A}$ does not satisfy $(PSC)_0$ and (5) if $p = 2$ then $\tilde{A}$ satisfies $(PS)_c$ for all $c > 0$.

As an application of our main theorem we study the Fučik spectrum with weights. This spectrum is defined as the set $\Sigma(m,n)$ of those $(\alpha,\beta) \in \mathbb{R}^2$ such that

$$-\Delta_p u = \alpha m(x)(u^+)^{p-1} - \beta n(x)(u^-)^{p-1} \text{ in } \Omega, \quad \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega \tag{4}$$

has a nontrivial solution. If we denote by $\Sigma^*(m,n)$ the set $\Sigma(m,n)$ amputed of the lines $\{\lambda^*(m)\} \times \mathbb{R}$ et $\mathbb{R} \times \{\lambda^*(n)\}$, we prove that for any $s > 0$, the line $\beta = s\alpha$ in the (α,β) plane intersects $\Sigma^*(m,n) \cap (\mathbb{R}^+ \times \mathbb{R}^+)$. Moreover the first point in this intersection is given by $\alpha(s) = c(m, sn)$, $\beta(s) = s\alpha(s)$.

We obtain in this way a first curve $\mathcal{C} := \{(\alpha(s), \beta(s)) : s > 0\}$ in $\Sigma^*(m,n) \cap (\mathbb{R}^+ \times \mathbb{R}^+)$.

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Antimaximum principle and Fučik spectrum

J.-P. Gossez (Bruxelles)

It is well-known that the antimaximum principle holds uniformly for the problem

$$\begin{cases} -u'' = \lambda u + h(x) & \text{on }]0,\pi[, \\ u'(0) = u'(\pi) = 0, \end{cases} \tag{5}$$

and that the interval of uniformity is $\lambda \in]0, 1/4[$. It is also well-known that the first curve in the Fučik spectrum for the problem

$$\begin{cases} -u'' = \alpha u^+ - \beta u^- & \text{on }]0,\pi[, \\ u'(0) = u'(\pi) = 0 \end{cases} \tag{6}$$

exhibits a gap at infinity with respect to the trivial horizontal and vertical lines, and that the value of this gap is equal to 1/4. When the Neumann conditions are replaced in (5) and (6) by the Dirichlet conditions, the antimaximum principle does not hold uniformly and there is no gap at infinity in the Fučik spectrum.

It is our purpose in this talk to survey some results which show that the above qualitative and quantitative correspondance between "uniformity of the antimaximum principle" and "gap at infinity in the Fučik spectrum" holds in various other situations (general elliptic operators, p-laplacian). However it does not hold anymore in general when weights are introduced.

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The Fredholm alternative for the p-Laplacian: bifurcation from infinity, existence and multiplicity

P. Drábek (Rostock), P. Girg (Plzeň), P. Takáč (Rostock)

We discuss the existence and multiplicity of solutions to the following boundary-value problem for the Dirichlet p-Laplacian in a bounded domain $\Omega \subset \mathbb{R}^N$:

$$\left\{ \begin{array}{rcll} -\Delta_p u - \lambda\, |u|^{p-2}\, u & = & f(x) & \text{in } \Omega\,; \\ u & = & 0 & \text{on } \partial\Omega\,. \end{array} \right. \tag{7}$$

Here, $\Delta_p u \stackrel{\text{def}}{=} \operatorname{div}(|\nabla u|^{p-2}\, \nabla u)$ where $p \in (1,\infty)$ is a fixed number, $f \in L^\infty(\Omega)$, and $\lambda \in \mathbb{R}$ is *spectral* parameter. Given $\lambda \in \mathbb{R}$, the solvability of (7) is closely related to the existence of a nontrivial solution of the corresponding eigenvalue problem

$$\left\{ \begin{array}{rcll} -\Delta_p u & = & \lambda\, |u|^{p-2}\, u & \text{in } \Omega\,; \\ u & = & 0 & \text{on } \partial\Omega\,, \end{array} \right. \tag{8}$$

which is nonlinear if $p \neq 2$ and linear for $p = 2$.

The first results applicable to the solvability of (7) go back to the works of Fučík et al. [10] and Pohozaev [11]: *If $\lambda \in \mathbb{R}$ is not an eigenvalue of* (8) *then* (7) *has at least one solution for any $f \in W^{-1,p'}(\Omega)$, $p = p/(p-1)$.*

Let $\lambda_1 > 0$ be the principal eigenvalue of $-\Delta_p$ subject to homogeneous Dirichlet boundary conditions. We concentrate on the behavior of the solutions under the assumption that λ is near λ_1 (and possibly $\lambda = \lambda_1$). Our main tool combines bifurcation theory and asymptotic estimates.

We first motivate our results by considering the linear boundary value problem

$$\left\{ \begin{array}{rl} -\Delta u - \lambda u = f & \text{in } \Omega\,; \\ u = 0 & \text{on } \partial\Omega\,, \end{array} \right. \tag{9}$$

which corresponds to $p = 2$ in (7). Let $f \in L^2(\Omega)$ be given, $f \not\equiv 0$. Then the set of all pairs $(\lambda, u) \in (-\infty, \lambda_2) \times W_0^{1,2}(\Omega)$ that satisfy (9) can be interpreted by means of a bifurcation diagram in $\mathbb{R} \times W_0^{1,2}(\Omega)$. Namely, let us write $u = c\varphi_1 + u^\top$ with $\int_\Omega u^\top \varphi_1 \, \mathrm{d}x = 0$. Here, φ_1 is the eigenfunction of the positive Dirichlet Laplacian $-\Delta$ associated with the (simple) principal eigenvalue λ_1 that is normalized by $\varphi_1 > 0$ in Ω and $\int_\Omega \varphi_1^2 \, \mathrm{d}x = 1$, and λ_2 stands for the second eigenvalue of $-\Delta$. Then problem (9) is equivalent to

$$\left\{ \begin{array}{rl} -\Delta u^\top - \lambda u^\top + (\lambda_1 - \lambda) c \varphi_1 = f^\top + a\varphi_1 & \text{in } \Omega\,; \\ u^\top = 0 & \text{on } \partial\Omega\,, \end{array} \right.$$

where $\int_\Omega f^\top \varphi_1 \, \mathrm{d}x = 0$ and $a = \int_\Omega f \varphi_1 \, \mathrm{d}x$. Clearly, $(\lambda_1 - \lambda)c = a$. The linear Fredholm alternative implies that the problem

$$\left\{ \begin{array}{rl} -\Delta u^\top - \lambda u^\top = f^\top & \text{in } \Omega\,; \\ u^\top = 0 & \text{on } \partial\Omega\,, \end{array} \right.$$

has a unique solution $u^\top \in W_0^{1,2}(\Omega)$ with $\int_\Omega u^\top \varphi_1 \,\mathrm{d}x = 0$. We have the following two different cases:

(i) If $\int_\Omega f\varphi_1 \,\mathrm{d}x = 0$ then
  (a) for any $\lambda \in (-\infty, \lambda_1) \cup (\lambda_1, \lambda_2)$, problem (9) has a unique solution $u_\lambda = u^\top$;
  (b) for $\lambda = \lambda_1$, all solutions of problem (9) can be written in the form $u_{\lambda_1} = c\varphi_1 + u^\top$ with $c \in \mathbb{R}$ arbitrary.

(ii) If $\int_\Omega f\varphi_1 \,\mathrm{d}x \neq 0$ then
  (a) there is no solution of (9) for $\lambda = \lambda_1$;
  (b) for any $\lambda \in (-\infty, \lambda_1) \cup (\lambda_1, \lambda_2)$ there is a unique solution of (9) expressed by $u_\lambda = c\varphi_1 + u^\top$ where

$$c = (\lambda_1 - \lambda)^{-1} \int_\Omega f\varphi_1 \,\mathrm{d}x.$$

The solution pairs $(\lambda, u) \in \mathbb{R} \times W_0^{1,2}(\Omega)$ of (9) can thus be sketched in the bifurcation diagrams indicated in Figure 1

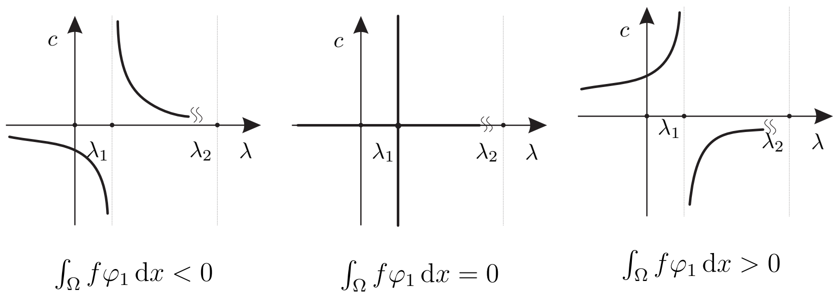

FIGURE 1. Bifurcations from infinity of solutions to (9), $c \stackrel{\text{def}}{=} \int_\Omega u\varphi_1 \,\mathrm{d}x$.

Motivated by this picture of the solution set of (9), we have decided to study the nonlinear problem (7) for $p \neq 2$ and to investigate the solution pairs $(\lambda, u) \in \mathbb{R} \times W_0^{1,p}(\Omega)$ for λ near λ_1. Again, φ_1 is the eigenfunction of the positive p-Laplacian associated with λ_1 and normalized by $\varphi_1 > 0$ and $\int_\Omega \varphi_1^p \,\mathrm{d}x = 1$. Notice that $a = \left(\int_\Omega \varphi_1^2 \,\mathrm{d}x\right)^{-1} \int_\Omega f\varphi_1 \,\mathrm{d}x$.

The existence of solutions $(\lambda, u) \in \mathbb{R} \times W_0^{1,p}(\Omega)$ to (7) with $\lambda \to \lambda_1$ and $\|u\|_{W_0^{1,p}(\Omega)} \to \infty$ is guaranteed by Dancer's type global bifurcation result for bifurcations from infinity at $\lambda = \lambda_1$. Roughly speaking, two continua $\mathcal{C}^\pm \subset R \times W_0^{1,p}(\Omega)$ of solutions to (7) emanate from (λ_1, ∞). Moreover, $\lambda \to \lambda_1$, $\|u\|_{W_0^{1,p}(\Omega)} \to \infty$ and $u \in \mathcal{C}^\pm$ imply $u/\|u\|_{W_0^{1,p}(\Omega)} \to \pm\varphi_1/\|\varphi_1\|_{W_0^{1,p}(\Omega)}$. If there is no sequence $\{(\lambda_1, u_n)\}_{n=1}^\infty$ of solutions to (7) such that $\|u_n\|_{W_0^{1,p}(\Omega)} \to \infty$, these two continua satisfy some very important global properties in addition; we refer to [4, 8] for a precise statement of this result.

We will establish an asymptotic estimate that plays the key role in the study of the structure of the solution set to (7). We assume $1 < p < \infty$, $p \neq 2$, if not explicitely mentioned otherwise. From now on, we denote by λ_2 ($\lambda_2 > \lambda_1$) the second eigenvalue of the positive Dirichlet p-Laplacian $-\Delta_p$. We use only the well-known fact from [2] that there is no eigenvalue of $-\Delta_p$ in the open interval (λ_1, λ_2), by a variational characterization of λ_2. All results presented here have been proved and reported in [8].

For the behavior of solutions u with large norm, the following a priori estimate plays the key role. We introduce some notation first. We introduce a new norm on $W_0^{1,p}(\Omega)$ by

$$\|v\|_{\mathcal{D}_{\varphi_1}} \stackrel{\text{def}}{=} \left(\int_\Omega |\nabla\varphi_1|^{p-2} |\nabla v|^2 \, \mathrm{d}x\right)^{1/2} \quad \text{for } v \in W_0^{1,p}(\Omega), \tag{10}$$

and denote by $\mathcal{D}_{\varphi_1}$ the completion of $W_0^{1,p}(\Omega)$ with respect to this norm. The Hilbert space $\mathcal{D}_{\varphi_1}$ is compactly imbedded in the Lebesgue space $L^2(\Omega)$; see [13, Lemma 4.2]. It is also shown there that the seminorm (10) is in fact a norm on $W_0^{1,p}(\Omega)$, if $2 < p < \infty$. For the case $1 < p < 2$ the space $\mathcal{D}_{\varphi_1}$ needs to be redefined. We do not need it for the formulation of any theorem here. Therefore its definition is omitted though it plays a key role in the proofs (see [8, 12, 13] for details).

For the sake of brevity, we also define

$$\mathcal{A}_{\varphi_1} \stackrel{\text{def}}{=} |\nabla\varphi_1|^{p-2} \left(\mathbf{I} + (p-2)\,\frac{\nabla\varphi_1 \otimes \nabla\varphi_1}{|\nabla\varphi_1|^2}\right) \quad \text{for } \nabla\varphi_1 \in \mathbb{R}^N,\ \nabla\varphi_1 \neq \mathbf{0} \in \mathbb{R}^N,$$

with $\mathbf{I}$ being the $n \times n$ identity matrix and $\otimes$ the tensor product.

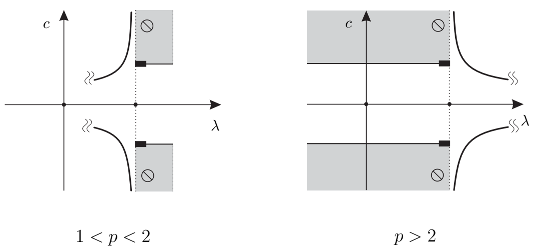

FIGURE 2. A priori bounds and bifurcations from infinity of solutions to (7) for $p > 1$, $p \neq 2$ and $a = 0$. There is no solution in the shaded regions (owing to a priori bounds).

Theorem 2. ([8, Thm. 4.1]) *Let* $\{\mu_n\}_{n=1}^\infty \subset \mathbb{R}$, $\{f_n\}_{n=1}^\infty \subset L^\infty(\Omega)$, $\{u_n\}_{n=1}^\infty \subset W_0^{1,p}(\Omega)$ *be sequences, and let* $\delta > 0$ *be such that*

(i) $\lambda_1 + \mu_n < \lambda_2 - \delta$ *for all* $n \in \mathbb{N}$*;*
(ii) $f_n \stackrel{*}{\rightharpoonup} f$ *weakly-star in* $L^\infty(\Omega)$*;*
(iii) $\|u_n\|_{W_0^{1,p}(\Omega)} \to \infty$ *as* $n \to \infty$*;*
(iv) *in addition, assume that for all* $n \in \mathbb{N}$ *and* $\phi \in W_0^{1,p}(\Omega)$,

$$\int_\Omega |\nabla u_n|^{p-2} \langle \nabla u_n, \nabla\phi\rangle \,\mathrm{d}x = (\lambda_1 + \mu_n) \int_\Omega |u_n|^{p-2} u_n\, \phi \,\mathrm{d}x + \int_\Omega f_n\, \phi\,\mathrm{d}x\,. \tag{11}$$

Then $\mu_n \to 0$ *and, writing* $u_n = t_n^{-1}(\varphi_1 + v_n^\top)$ *with* $t_n \in \mathbb{R}$, $t_n \neq 0$, *and* $v_n^\top \in W_0^{1,p}(\Omega)^\top$, *we have* $t_n \to 0$, $|t_n|^{-p}\, t_n\, v_n^\top \to V^\top$ *strongly in* $\mathcal{D}_{\varphi_1}$ *if* $p > 2$ *and in* $W_0^{1,2}(\Omega)$ *if* $1 < p < 2$, *and*

$$\begin{aligned}\mu_n = &-|t_n|^{p-2}\, t_n \int_\Omega f_n\, \varphi_1 \,\mathrm{d}x + (p-2)\, |t_n|^{2(p-1)}\ \mathcal{Q}_0(V^\top, V^\top) \\ &+ (p-1)\, |t_n|^{2(p-1)} \left(\int_\Omega f\, \varphi_1\,\mathrm{d}x\right) \left(\int_\Omega \varphi_1^{p-1}\, V^\top\,\mathrm{d}x\right) + o\left(|t_n|^{2(p-1)}\right).\end{aligned} \tag{12}$$

In particular, if $\int_\Omega f_n\, \varphi_1\,\mathrm{d}x = 0$ *for all* $n \in \mathbb{N}$, *then*

$$\mu_n = (p-2)\, |t_n|^{2(p-1)}\ \mathcal{Q}_0(V^\top, V^\top) + o\left(|t_n|^{2(p-1)}\right).$$

Moreover, $V^\top \in \mathcal{D}_{\varphi_1} \cap \{\varphi_1\}^{\perp, L^2}$ *is the (unique) solution to*

$$2\cdot \mathcal{Q}_0(V^\top, \phi) = \int_\Omega f^\dagger\, \phi\,\mathrm{d}x \quad \text{for all } \phi \in \mathcal{D}_{\varphi_1}, \tag{13}$$

where we have denoted

$$2\cdot \mathcal{Q}_0(V^\top, \phi) = \int_\Omega \left\langle \mathbf{A}_{\varphi_1} \nabla V^\top,\, \nabla\phi\right\rangle \mathrm{d}x - \lambda_1\,(p-1) \int_\Omega \varphi_1^{p-2}\, V^\top\, \phi\,\mathrm{d}x$$

and $f^\dagger = f - \left(\int_\Omega f\, \varphi_1\,\mathrm{d}x\right) \varphi_1^{p-1}$.

Remark 1. The linear equation (13) represents the weak form of the "limiting" Dirichlet boundary value problem for the limit function $|t_n|^{-p}\, t_n\, v_n^\top \to V^\top$ in the approximation scheme with $u_n = t_n^{-1}(\varphi_1 + v_n^\top)$. This is a resonant problem to which a standard version of the Fredholm alternative for a selfadjoint linear operator in a Hilbert space applies. More precisely, given a function $f \in L^2(\Omega)$, a weak solution $V \in \mathcal{D}_{\varphi_1}$ to the equation

$$2\cdot \mathcal{Q}_0(V, \phi) = \textstyle\int_\Omega f\, \phi\,\mathrm{d}x \quad \text{for all } \phi \in \mathcal{D}_{\varphi_1}, \tag{14}$$

exists in $\mathcal{D}_{\varphi_1}$ if and only if $\int_\Omega f\, \varphi_1\,\mathrm{d}x = 0$. Such a solution is always unique under the orthogonality condition $\int_\Omega V\, \varphi_1\,\mathrm{d}x = 0$.

Note that (14) written in divergent form reads as follows (see e.g. [8, 12, 13])

$$\begin{aligned}\operatorname{div}(\mathbf{A}_{\varphi_1} \nabla V^\top) - \lambda_1 \varphi_1^{p-2} V^\top &= f \quad \text{in } \Omega\,;\\ V^\top &= 0 \quad \text{on } \partial\Omega\,;\\ \int_\Omega V^\top \varphi_1 &= 0\,.\end{aligned}$$

Remark 2. In fact we also use a variant of Theorem 2 (see [8, Cor. 4.4] for details) in order to prove the following uniform result.

Let K be a closed bounded ball in $L^\infty(\Omega)$. Assume that $f_n \equiv f$ $(n = 1, 2, \dots)$ and $t_n \to 0$ as $n \to \infty$. Then there exists a sequence $\{\eta_n\}_{n=1}^\infty \subset (0,1)$, $\eta_n \to 0$ as $n \to \infty$, such that for all $f \in K$ and for all $n = 1, 2, \dots$ we have

$$\begin{aligned} \Bigg| |t_n|^{-2(p-1)} \left(\mu_n - |t_n|^{p-2}\, t_n \int_\Omega f\, \varphi_1 \,\mathrm{d}x \right) - (p-2) \cdot \mathcal{Q}_0(V^\top, V^\top) \\ - (p-1) \left(\int_\Omega f\, \varphi_1 \,\mathrm{d}x \right) \left(\int_\Omega \varphi_1^{p-1}\, V^\top \,\mathrm{d}x \right) \Bigg| \le \eta_n. \end{aligned} \tag{15}$$

The main results concerning the asymptotic behavior of the solution set to (7) as $\lambda \to \lambda_1$ are sketched in Figures 2 and 3. We assume that $f^\top \in L^\infty(\Omega)$ is a given function satisfying $\int_\Omega f^\top \varphi_1 \,\mathrm{d}x = 0$ and $f^\top \not\equiv 0$. In (7) we write $f = a\varphi_1 + f^\top$, $a \in \mathbb{R}$, and split the solution as $u = c\varphi_1 + u^\top$. Note, that there are no solutions in the shaded regions (we have a priori bounds) while there may be many other solutions in the nonshaded regions.

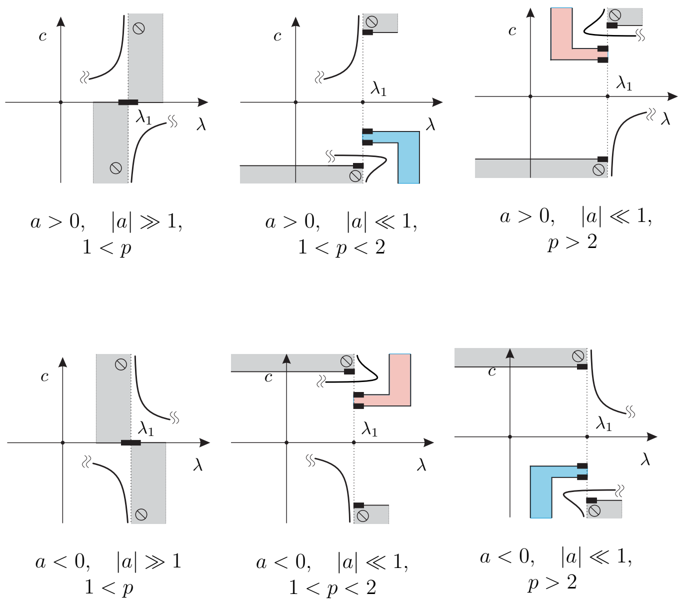

FIGURE 3. A priori bounds and bifurcations from infinity of solutions to (7) for $a \neq 0$, $1 < p < 2$ and/or $p > 2$.

We rewrite problem (7) as follows, with $f = f^\top + a\varphi_1$:

$$\begin{cases} -\Delta_p u - \lambda\, |u|^{p-2}\, u = f^\top + a\varphi_1 & \text{in } \Omega\,; \\ u = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{16}$$

Here, $f^\top \in L^\infty(\Omega)$ is a given function, with $\int_\Omega f^\top \varphi_1 \,\mathrm{d}x = 0$ and $f^\top \not\equiv 0$, and $\lambda, a \in \mathbb{R}$ are real parameters. We split the solution as $u = c\varphi_1 + u^\top$. Basic multiplicity results are obtained from the shape of the continua emanating from (λ_1, ∞). Additional multiplicity results are deduced from the shape of the continua using the method of upper and lower solutions. For the convenience of the reader, we organize these results in following two tables. Dependence of the existence, multiplicity and *a priori* bounds of the solutions on the spectral parameter λ can easily be deduced from these tables.

Let us note that the theory developped in [8] can be used in the study of a more general boundary value problem

$$-\Delta_p u - \lambda |u|^{p-2} u = h(u, x) \text{ in } \Omega \quad u = 0 \text{ on } \partial\Omega\,. \tag{17}$$

Interested reader is refered to [7].

Finally, we would also like to note that the strongly nonlinear boundary value problems emphasize the importance of the interplay between numerical experiments and development of new theoretical methods, see e.g. [3].

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$1 < p < 2,\ f \not\equiv 0$

$\lambda < 0$	$0 \le \lambda \le \lambda_1$	λ_1	$\lambda_1 < \lambda < \lambda_2 - \delta$
$\Vert u\Vert_{C^{1,\beta(\overline{\Omega})}} \le M(f^\top, a, p, \lambda, \Omega) < \infty$			
$M(f^\top, a, p, \lambda, \Omega) \to \infty$ as $\lambda \to \lambda_{1-}$		$a = 0 \Rightarrow M(f^\top, a, p, \lambda, \Omega) < M < \infty \quad \forall\lambda$ $M(f^\top, a, p, \lambda_1, \Omega) < \infty \quad \forall a \in \mathbb{R}$	$a \neq 0 \Rightarrow M(f^\top, a, p, \lambda, \Omega) \to \infty$ as $\lambda \to \lambda_{1+}$
$\mathsf{S} = 1$	$\mathsf{S} \ge 1$		$\mathsf{S} \ge 1$
	$f \prec 0$ $\quad \mathsf{S}, \mathsf{N} = 1, \mathsf{P} = 0$ *(LMP)*	$f \prec 0$ or $a < \underline{A}$ $\quad \mathsf{S} = 0$	$f \prec 0$ or $a < \underline{A}$ *(LAMP)* $\exists\eta > 0\,\forall\lambda \in (\lambda_1, \lambda_1 + \eta)$: $\mathsf{P} \ge 1, \mathsf{S} - \mathsf{P} = 0$
	$a < \underline{A}$ *(LMP)* $\exists\eta > 0\,\forall\lambda \in (\lambda_1 - \eta, \lambda_1)$: $\mathsf{N} \ge 1, \mathsf{S} - \mathsf{N} = 0$		
	$\underline{A} < a < \underline{a}$ $\quad \mathsf{N} \ge 1$ PAB	$\underline{A} < a < \underline{a}$ $\quad \mathsf{S} \ge 2$ *(UpLow)*	$\underline{A} < a < \underline{a}$ $\quad \mathsf{P} \ge 1$, NAB
	$\forall\varepsilon' \in (0, -\underline{a})\exists\eta = \eta(f^\top, \underline{a}, \varepsilon')\ \forall\lambda$: $\lambda_1 - \eta < \lambda < \lambda_1$, $\underline{a} < a < -\varepsilon'$ $\quad \mathsf{S} \ge 3, \mathsf{P}, \mathsf{N} \ge 1$	$\underline{a} < a < 0$ $\quad \mathsf{S} \ge 2, \mathsf{P} \ge 1$; $M(f^\top, a, p, \Omega) \to \infty$ as $a \to 0_-$	$\forall\varepsilon' \in (0, -\underline{a})\exists\eta = \eta(f^\top, \underline{a}, \varepsilon')\ \forall\lambda$: $\lambda_1 < \lambda < \lambda_1 + \eta$, $\underline{a} < a < -\varepsilon'$ $\quad \mathsf{S} \ge 2, \mathsf{P} \ge 2$
	$\lambda_1 - \eta < \lambda < \lambda_1$, $a = 0$ $\quad \mathsf{S} \ge 3, \mathsf{P}, \mathsf{N} \ge 1$	$a = 0$ $\quad \mathsf{S} \ge 1$; $\Vert u\Vert_{C^{1,\beta(\overline{\Omega})}} \le M(f^\top, 0, p, \Omega) < \infty$	$\lambda_1 < \lambda < \lambda_1 + \eta$, $a = 0$ $\quad \mathsf{S} \ge 1$
	$\forall\varepsilon' \in (0, \overline{a})\exists\eta = \eta(f^\top, \overline{a}, \varepsilon')\ \forall\lambda$: $\lambda_1 - \eta < \lambda < \lambda_1$, $\varepsilon' < a < \overline{a}$ $\quad \mathsf{S} \ge 3, \mathsf{P}, \mathsf{N} \ge 1$	$0 < a < \overline{a}$ $\quad \mathsf{S} \ge 2, \mathsf{N} \ge 1$; $M(f^\top, a, p, \Omega) \to \infty$ as $a \to 0_+$	$\forall\varepsilon' \in (0, \overline{a})\exists\eta = \eta(f^\top, \overline{a}, \varepsilon')\ \forall\lambda$: $\lambda_1 < \lambda < \lambda_1 + \eta$, $\varepsilon' < a < \overline{a}$ $\quad \mathsf{S} \ge 2, \mathsf{N} \ge 2$
	$\overline{a} < a < \overline{A}$ $\quad \mathsf{P} \ge 1$, NAB	$\overline{a} < a < \overline{A}$ $\quad \mathsf{S} \ge 2$ *(UpLow)*	$\overline{a} < a < \overline{A}$ $\quad \mathsf{N} \ge 1$, PAB
	$\overline{A} < a$ *(LMP)* $\exists\eta > 0\,\forall\lambda \in (\lambda_1 - \eta, \lambda_1)$: $\mathsf{P} \ge 1, \mathsf{S} - \mathsf{P} = 0$	$\overline{A} < a$ or $f \succ 0$ $\quad \mathsf{S} = 0$	$\overline{A} < a$ or $f \prec 0$ *(LAMP)* $\exists\eta > 0\,\forall\lambda \in (\lambda_1, \lambda_1 + \eta)$: $\mathsf{N} \ge 1, \mathsf{S} - \mathsf{N} = 0$
	$f \succ 0$ $\quad \mathsf{S}, \mathsf{P} = 1, \mathsf{N} = 0$ *(LMP)*		

Legend: S (N, P) number of (negative, positive) solution; $\mathsf{S} \ge n$ at least n solutions; $\mathsf{N} \ge 1$ at least one negative solution; $\mathsf{S} - \mathsf{N} = 0$ all solutions are negative; NAB (PAB) negative (positive) solutions are *a priori* bounded; *(LMP)* Local Maximum Principle, *(LAMP)* Local Anti-Maximum Principle, *(UpLow)* by upper and lower solutions argument.

$p > 2,\ f \not\equiv 0$

$\lambda < 0$	$0 \le \lambda \le \lambda_1$	λ_1	$\lambda_1 < \lambda < \lambda_2 - \delta$
$\Vert u \Vert_{C^{1,\beta}(\overline{\Omega})} \le M(f^\top, a, p, \lambda, \Omega) < \infty$			
	$a = 0 \Rightarrow M(f^\top, a, p, \lambda, \Omega) < M < \infty \quad \forall \lambda$ $a \ne 0 \Rightarrow M(f^\top, a, p, \lambda, \Omega) \to \infty$ as $\lambda \to \lambda_{1-}$	$M(f^\top, a, p, \lambda_1, \Omega) < \infty \quad \forall a \in \mathbb{R}$	$M(f^\top, a, p, \lambda, \Omega) \to \infty$ as $\lambda \to \lambda_{1+}$
	$\mathsf{S} \ge 1$		$\mathsf{S} \ge 1$
	$f \prec 0$ $\mathsf{S}, \mathsf{N} = 1, \mathsf{P} = 0$ *(LMP)*	$f \prec 0$ or $a < \underline{A}$ $\mathsf{S} = 0$	$f \prec 0$ or $a < \underline{A}$ *(LAMP)* $\exists \eta > 0 \, \forall \lambda \in (\lambda_1, \lambda_1 + \eta):$ $\mathsf{P} \ge 1$, $\mathsf{S} - \mathsf{P} = 0$
	$a < \underline{A}$ *(LMP)* $\exists \eta > 0 \, \forall \lambda \in (\lambda_1 - \eta, \lambda_1):$ $\mathsf{N} \ge 1, \mathsf{S} - \mathsf{N} = 0$		
	$\underline{A} < a < \underline{a}$ $\mathsf{N} \ge 1$ PAB	$\underline{A} < a < \underline{a}$ $\mathsf{S} \ge 2$ *(UpLow)*	$\underline{A} < a < \underline{a}$ $\mathsf{P} \ge 1$, NAB
	$\forall \varepsilon' \in (0, -\underline{a}) \exists \eta = \eta(f^\top, \underline{a}, \varepsilon') \quad \forall \lambda:$ $\lambda_1 - \eta < \lambda < \lambda_1$ $\underline{a} < a < -\varepsilon'$ $\mathsf{S} \ge 2, \mathsf{N} \ge 2$	$\underline{a} < a < 0$ $\mathsf{S} \ge 2$, $\mathsf{P} \ge 1$ $M(f^\top, a, p, \Omega) \to \infty$ as $a \to 0_-$	$\forall \varepsilon' \in (0, -\underline{a}) \exists \eta = \eta(f^\top, \underline{a}, \varepsilon') \quad \forall \lambda:$ $\lambda_1 < \lambda < \lambda_1 + \eta$ $\mathsf{S} \ge 3, \mathsf{P}, \mathsf{N} \ge 1$
$\mathsf{S} = 1$	$\lambda_1 - \eta < \lambda < \lambda_1$ $a = 0$ $\mathsf{S} \ge 1$	$a = 0$ $\mathsf{S} \ge 1$ $\Vert u \Vert_{C^{1,\beta}(\overline{\Omega})} \le M(f^\top, 0, p, \Omega) < \infty$	$\lambda_1 < \lambda < \lambda_1 + \eta$ $a = 0$ $\mathsf{S} \ge 3, \mathsf{P}, \mathsf{N} \ge 1$
	$\forall \varepsilon' \in (0, \overline{a}) \exists \eta = \eta(f^\top, \overline{a}, \varepsilon') \quad \forall \lambda:$ $\lambda_1 - \eta < \lambda < \lambda_1$ $\varepsilon' < a < \overline{a}$ $\mathsf{S} \ge 2$, $\mathsf{P} \ge 2$	$0 < a < \overline{a}$ $\mathsf{S} \ge 2$, $\mathsf{N} \ge 1$ $M(f^\top, a, p, \Omega) \to \infty$ as $a \to 0_+$	$\forall \varepsilon' \in (0, \overline{a}) \exists \eta = \eta(f^\top, \overline{a}, \varepsilon') \quad \forall \lambda:$ $\lambda_1 < \lambda < \lambda_1 + \eta$ $\varepsilon' < a < \overline{a}$ $\mathsf{S} \ge 3, \mathsf{P}, \mathsf{N} \ge 1$
	$\overline{a} < a < \overline{A}$ $\mathsf{P} \ge 1$, NAB	$\overline{a} < a < \overline{A}$ $\mathsf{S} \ge 2$ *(UpLow)*	$\overline{a} < a < \overline{A}$ $\mathsf{N} \ge 1$, PAB
	$\overline{A} < a$ *(LMP)* $\exists \eta > 0 \, \forall \lambda \in (\lambda_1 - \eta, \lambda_1):$ $\mathsf{P} \ge 1, \mathsf{S} - \mathsf{P} = 0$	$\overline{A} < a$ or $f \succ 0$ $\mathsf{S} = 0$	$\overline{A} < a$ or $f \prec 0$ *(LAMP)* $\exists \eta > 0 \, \forall \lambda \in (\lambda_1, \lambda_1 + \eta):$ $\mathsf{N} \ge 1$, $\mathsf{S} - \mathsf{N} = 0$
	$f \succ 0$ $\mathsf{S}, \mathsf{P} = 1, \mathsf{N} = 0$ *(LMP)*		

Legend: S (N, P) number of (negative, positive) solution; $\mathsf{S} \ge n$ at least n solutions; $\mathsf{N} \ge 1$ at least one negative solution; $\mathsf{S} - \mathsf{N} = 0$ all solutions are negative; NAB (PAB) negative (positive) solutions are *a priori* bounded; *(LMP)* Local Maximum Principle, *(LAMP)* Local Anti-Maximum Principle, *(UpLow)* by upper and lower solutions argument.

Perturbation of the simple eigenvalue by 1-homogeneous operators
Raffaele Chiappinelli (Siena, Italy)

Let T be a bounded linear operator acting in a real Banach space E and suppose that T has an isolated eigenvalue of finite multiplicity λ_0. If we add to T a perturbation term εB, with B (positively) homogeneous of degree 1, continuous and such that $B(0) = 0$, then we ask the following questions:

1) Do we find eigenvalues of $T + \varepsilon B$ near λ_0?

2) If this is the case, are these eigenvalues isolated themselves?

(An *eigenvalue* of an operator $F : E \to E$ such that $F(0) = 0$ is a $\lambda \in \mathbb{R}$ such that $F(u_0) = \lambda u_0$ for some *eigenvector* $u_0 \neq 0$; in this case we say that

$$N(F - \lambda I) \equiv \{u \in E : F(u) - \lambda u = 0\}$$

is the *eigenset* corresponding to λ. If F is (1-)homogeneous, this notion of eigenvalue coincides with that of *connected eigenvalue* proposed in [4]).

Simple examples in finite dimension show the answer to both questions may be "No". In particular, as for question 2) one may consider the equation

$$x + \varepsilon\phi\Big(\frac{x}{\|x\|}\Big)x = \lambda x, \qquad x \in \mathbb{R}^N \tag{18}$$

where $\phi : S \equiv \{x \in \mathbb{R}^N : \|x\| = 1\} \to \mathbb{R}$ is continuous. Then $Tx \equiv x$, $B(x) \equiv \phi(\frac{x}{\|x\|})x$ for $x \neq 0$, $B(0) = 0$ satisfy the above assumptions. However, each $x \in S$ is an eigenvector of (18) corresponding to the eigenvalue $\lambda = \lambda(x) = 1 + \varepsilon\phi(x)$; thus if $N > 1$, then - as S is connected in this case - $\{1 + \varepsilon\phi(x) : x \in S\}$ is an *interval* of eigenvalues (except when ϕ is constant on S) which for ε small is close as we wish to the "unperturbed" eigenvalue 1 of T.

On the other hand, it is possible to provide an affirmative answer when λ_0 is (algebraically) *simple* and B is Lipschitz continuous: indeed, in this case we essentially prove that λ_0 splits (for each $\varepsilon \neq 0$) into precisely two eigenvalues $\lambda_\pm(\varepsilon)$, while the eigenline $N(T - \lambda_0 I)$ correspondingly "breaks" into two eigenrays $N_\pm(\varepsilon)$. For the Hilbert space case, the precise statement is as follows:

Theorem 3. *Let T be a bounded linear operator in H (a real Hilbert space with scalar product denoted by $\langle,\rangle$), and let $B : H \to H$ be such that $B(0) = 0$. Suppose that:*

(i) *T is selfadjoint and λ_0 is an isolated and simple eigenvalue of T;*
(ii) *B is Lipschitz continuous of constant k;*
(iii) *B is homogeneous.*

Then there exist $\delta_0 > 0, \varepsilon_0 > 0$ (depending only on λ_0 and k) such that for every ε with $|\varepsilon| \leq \varepsilon_0$, $T + \varepsilon B$ has precisely two *(possibly coinciding) eigenvalues $\lambda_+(\varepsilon), \lambda_-(\varepsilon)$ in the interval $[\lambda_0 - \delta_0, \lambda_0 + \delta_0]$: that is, for $|\lambda - \lambda_0| \leq \delta_0$ nontrivial solutions of the equation*

$$Tu + \varepsilon B(u) = \lambda u \tag{19}$$

exist if and only if $\lambda = \lambda_\pm(\varepsilon)$. *Moreover, the eigensets* $N_\pm(\varepsilon) \equiv N(T + \varepsilon B - \lambda_\pm(\varepsilon)I)$ *corresponding to* $\lambda_\pm(\varepsilon)$ *are rays in* H, *that is, there exist nonzero vectors* $z_\pm(\varepsilon) \in H$ *such that*

$$N_\pm(\varepsilon) = \{tz_\pm(\varepsilon) : t \geq 0\}.$$

Finally $\lambda_\pm(\varepsilon)$ *and* $z_\pm(\varepsilon)$ *depend Lipschitz-continuously upon* ε *for* $|\varepsilon| \leq \varepsilon_0$, *and if* ϕ *is a normalized eigenvector of* T *corresponding to* λ_0, *then as* $\varepsilon \to 0$ $z_\pm(\varepsilon) \to \pm\phi$ *and*

$$\lambda_\pm(\varepsilon) = \lambda_0 + \varepsilon\langle B(\pm\phi), \pm\phi\rangle + o(\varepsilon).$$

Theorem 3 is proved in [1] by first using the Lyapounov-Schmidt reduction for (19), and then making full use of the homogeneity of B in the resulting bifurcation equation. In a sense, this generalizes a result of Ruf [3] concerning the existence and uniqueness of two eigenvalues $\mu_k^1, \mu_k^2 \in [\mu_k^0, \mu_{k+1}^0[$ for the problem (in a bounded open set $\Omega \subset \mathbb{R}^N$)

$$Lu = \gamma u^- + \mu u \quad \text{in} \quad \Omega, \qquad u = 0 \quad \text{on} \quad \partial\Omega \tag{20}$$

with L linear elliptic selfadjoint and $u^- = \max(-u, 0)$, near each simple eigenvalue μ_k^0 of L. In fact, similar results hold (see [1]) for the problem

$$Lu = \mu(u + \varepsilon(\alpha(x)u^+ - \beta(x)u^-)) \quad \text{in} \quad \Omega, \qquad u = 0 \quad \text{on} \quad \partial\Omega \tag{21}$$

with $u = u^+ - u^-$ and $\alpha, \beta \in L^\infty(\Omega)$. Moreover, in the special case $\alpha, \beta = const$ we obtain informations about the structure of the "Fučik spectrum" Σ of L near (μ_k^0, μ_k^0), which agree with classical results [2].

Open problem: Describe what happens when λ_0 is not simple. Also, single out a class of homogeneous mappings in $\mathbb{R}^N$ all of whose eigenvalues are isolated (as for linear mappings).

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Remarks on some inhomogeneus eigenvalue problems

Vesa Mustonen (Oulu)

We discuss the "principal" eigenvalues of the problem

$$\begin{cases} -\Delta_m(u) = \lambda m(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{22}$$

where $m : [0,\infty) \mapsto [0,\infty)$ is nondecreasing continuous function with $m(0) = 0$, $m(t) > 0$ and $t > 0$, $\lim_{t\to\infty} = \infty$, $m(-t) = -m(t)$ $\forall t \in \mathbb{R}^n$, $\Omega \subset \mathbb{R}^n$ bounded open subset and

$$\Delta_m(u) := \operatorname{div}\left(\frac{m(|\nabla u|)}{|\nabla u|}\nabla u\right) \qquad \text{(generalized Laplacian).}$$

It is known ([2], [1]) that for each $r > 0$ the solutions $u_r \in W_0^1 L_M(\Omega)$ of the minimization problem

$$\inf\left\{\int_\Omega M(|\nabla u|) : u \in W_0^1 L_M(\Omega),\ \int_\Omega M(u) = r\right\}$$

are solutions for (22) with some $\lambda = \lambda_r > 0$. (Here $M(t) = \int_0^t m(s)ds$). Some examples for the ODE-case suggest that the set of "principal" eigenvalues $\lambda > 0$ obtained is not necessarily bounded from above or bounded from below away from zero ([3]) Therefore we suggest to modify the problem (22) to the form

$$\begin{cases} -\Delta_m(u) = \lambda m(\lambda u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{23}$$

For the ODE-case

$$\begin{cases} -(m(u'))' = \lambda m(\lambda u) & \text{in } (0,a) \\ u(0) = u(a) = 0 \end{cases} \tag{24}$$

one can use the time map which suggests that all principal eigenvalues for (24) are in the bounded interval $[2/a, 4/a]$. This is joint work with Matti Tienari, University of Oulu /Central Laboratory, Helsinki.

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Applications of the degree for Fredholm maps to elliptic problems

C. A. Stuart (Lausanne)

A topological degree for C^1-Fredholm maps of index zero that are proper on closed bounded sets, has been defined by several approaches in a way that makes it possible to track the change in the degree under homotopy. See the work by Fitzpatrick, Pejsachowicz and Rabier [3,4,7] and then by Benevieri and Furi [1,2]. For the case of a map $F : X \to Y$ acting between two real Banach spaces X and Y, the following properties of the F are required.

(1) $F \in C^1(X, Y)$
(2) $F(u) \in B(X, Y)$ is a Fredholm operator of index zero for all $u \in X$
(3) $F : W \to Y$ is proper for all closed bounded subsets W of X.

In a series of papers written in collaboration with H. Jeanjean, M. Lucia, P. J. Rabier, S. Secchi and H. Gebran, we have used this degree to obtain results about the existence and bifurcation of solutions of systems of differential equations in several situations where the Leray-Schauder degree is not directly applicable. My lecture started with a summary of this work and then I presented in more detail the treatment of quasilinear systems that are elliptic in the sense of Petrovskii. I illustrated one of the keys steps in the case of a simple but typical example of a second order quasilinear elliptic equation.

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On the sign-jump of one-parameter families of Fredholm operators and bifurcation

Massimo Furi (Florence, Italy)

In [1] (see also [2] for more details) a fairly simple notion of orientation for Fredholm linear operators of index zero between real vector spaces was introduced. Any such operator, invertible or noninvertible, admits exactly two orientations, and the choice of an orientation makes, by definition, the operator oriented. However, if an operator is invertible, one of the two orientations is more relevant than the other, and for this reason called *natural*. Thus it makes sense to assign to any oriented isomorphism a sign: 1 if the orientation is natural and -1 in the opposite case. For a singular Fredholm operator of index zero no one of the two orientations is more relevant than the other.

A crucial fact is that in the framework of Banach spaces the orientation has a sort of stability; in the sense that an orientation of an operator L induces, in a very natural way, an orientation to any operator which is sufficiently close to L. Using this fact, the notion of orientation was extended (in [1]) to the nonlinear case; namely, to the case of a C^1 Fredholm map of index zero between real Banach spaces (and Banach manifolds). Such an extension coincides (in the C^1 case) with the notion given by Dold in [4, exercise 6, p. 271] for maps between finite dimensional manifolds and, in the most important cases, with the notion due to Fitzpatrick, Pejsachowicz and Rabier for maps between Banach manifolds (see [5–9]).

In [1], by means of the concept of orientation, a degree theory for Fredholm maps between Banach manifolds was introduced. This theory is purely based on the Brouwer degree, and in the most important cases agrees with the theory developed by Fitzpatrick, Pejsachowicz and Rabier in a series of papers ranging from 1991 to 1998. The difference between the two theories is mainly in the construction method and in a different definition of orientation.

This talk is inspired by a recent joint work with Benevieri, Pera and Spadini (see [3]), and it concerns methods for computing the degree by counting the sign-jumps in a continuous curve of Fredholm operators of index zero.

Some consequences in global bifurcation theory are derived from the detection of a sign-jump.

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Applications of nonlinear and semilinear spectral theory to boundary value problems

Wenying Feng (Peterborough, Canada)

We study the nonlinear spectrum $\sigma(f)$ and semilinear spectrum $\sigma(L, N)$, when L is Fredholm of index zero, f and N are asymptotically linear or positively homogeneous, thus close to a linear operator. The results generalize a previous result which required N to be a linear operator and L to be the identity map. To prove a theorem on the spectrum of asymptotically linear operator, we introduce the field of regularity for semilinear operators. When N is a positively homogeneous operator, we give a condition that ensures the existence of a positive eigenvalue for the semilinear pair (L, N).

The theorems can be applied to the study of some integral equations involving Urysohn and Hammerstein operators. Results on existence of solutions, bifurcation points, asymptotic bifurcation points are obtained. We also apply our theorems to the study of the second order differential equation:

$$u'' + f(t, u) = 0 \tag{25}$$

with one of the boundary conditions ($0 < \eta < 1$ fixed):

$$x(0) = 0,\ x(1) = \alpha x(\eta), \tag{26}$$

$$x'(0) = 0,\ x(1) = \alpha x(\eta). \tag{27}$$

By making use of a upper bound that involves the parameters α, η, we prove results on the existence of a solution, which in some cases are better than previous results (required a constant upper bound of f) of Gupta, Ntouyas and Tsamatos. Some examples show that there are equations that can be treated by our theorems but the previous results can not be applied. Moreover, with the assumption that f is positively homogeneous, we study the existence of an eigenvalue for the more general equation

$$u'' = f(t, u, u'),\ t \in (0, 1) \tag{28}$$

with one of the boundary condition (26) and (27). We give an alternative condition for existence of a positive eigenvalue and being a surjective map. Two examples are constructed to show that there are equations that satisfy our condition and so existence of an eigenvalue can be proved.

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Epi and Coepi Maps, and Further?

Martin Väth

This is a survey talk on the current state and possible developments of topological methods for coincidence points of function pairs which is one of the crucial ingredients of nonlinear spectral theory.

On the one hand, the concept of 0-epi maps (see e.g. [3]) can be considered as a definition of a homotopically stable coincidence point of two functions. On the other hand, there exist various degree theories for coincidence points which might be considered as a corresponding homologic approach: Degree theories for coincidence points of compact maps with linear Fredholm maps of zero or positive index [8, 9], with nonlinear Fredholm maps of index zero [2], or with monotone maps [11]. The link between these two kind of approaches (0-epi maps and degree theory) can be established by the famous Hopf theorems [5].

A third approach to coincidence points is given by various fixed point indices of multivalued maps (each of these indices is based on one of four essentiallz different ideas [1, 6, 7, 10] which are briefly sketched in the talk). This index approach is somewhat dual to the above coincidence degree theories and might be considered as an application of cohomology theory. It is possible to give a corresponding cohomotopic definition of a "coepi" concept which relates to these index theories by Hopf type theorems [13]. So, roughly, one has the following picture:

homotopic	homologic
Epi Maps Homotopy	*Degree* Homology
Coepi Maps Cohomotopy	*Index* Cohomology

All these concepts and Hopf theorems generalize also to noncompact but only condensing functions pairs. Moreover, it seems now that there is a natural notion of a degree for function triples which covers and extends all the above theories in a unified manner [4, 12].

As this is a survey talk, it would be too long to give a complete list of references in this abstract: For each referred subject only the historically first paper dealing with that topic is cited here.

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Spectral theory for homogeneous operators: part I

Elena Giorgieri (Rome)

The aim of this talk is to present a part of a joint work with J. Appell and M. Väeth, contained in the paper *Nonlinear spectral theory for homogeneous operators* [2], where, starting from the work [4], [3], [5], [6], and [1], we develop a parallel theory of spectra and phantoms which better describes properties of homogeneous operators of general degree. The above papers, with the exception of [5] and [6], deal with an operator F acting on a Banach space E and define the spectra by using some metric and topological characteristics and some notion of *solvability* of the equation

$$(\lambda I - F)(u) = G(u),$$

where $G : X \to Y$ varies in a suitable subset of the space of continuous operators. Moreover, the spectra introduced in [4] and in [1] depend essentially on the asymptotic properties of the operators involved and do not contain the eigenvalues (in the classical sense), while the spectrum in [3] is an example of "global" spectrum, because it is meant to contain all the eigenvalues. Regarding the papers [5] and [6], they deal with operators acting between two different Banach spaces and the spectra they define, called phantoms, describe the "local" behaviour of the operator. One of the main features of their work is the introduction of a new notion of eigenvalue for a pair of operators (F, J), where J replaces the identity (F acts between two different Banach spaces).

The basic idea of our work [2] is to modify the definitions given in the above papers in a way that takes into account the special behaviour of homogeneous operators. Indeed, we deal with *continuous* operators $F, J : X \to Y$ acting between two different Banach spaces X,Y (over the same field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) and satisfying $F(\theta) = J(\theta) = \theta$. Here J is some "well-behaved" operator that replaces the role of the identity, for example a homeomorphism, while F denotes the operator we want to analyse. The modified metric and topological characteristics we use are then the following.

Metric characteristics

$$M_\tau(F) = \sup_{u\neq\theta} \frac{\|F(u)\|}{\|u\|^\tau}, \qquad m_\tau(f) = \inf_{u\neq\theta} \frac{\|F(u)\|}{\|u\|^\tau},$$

$$|F|_\tau = \limsup_{\|u\|\to\infty} \frac{\|F(u)\|}{\|u\|^\tau}, \qquad d_\tau(F) = \liminf_{\|u\|\to\infty} \frac{\|F(u)\|}{\|u\|^\tau}, \qquad \tau > 0$$

Topological characteristics

$$\alpha_\tau(F) = \inf \left\{ \begin{array}{l} L \geq 0 : \alpha(F(M)) \leq L\alpha(M)^\tau \\ \text{for all bounded } M \subset X \end{array} \right\},$$

$$\beta_\tau(F) = \sup \left\{ \begin{array}{l} \ell \geq 0 : \alpha(F(M)) \geq \ell\alpha(M)^\tau \\ \text{for all bounded } M \subset X \end{array} \right\}, \qquad \tau > 0.$$

($\alpha(M)$ is the usual Kuratowski measure of noncompactness of the bounded subset M).

By adapting the definitions in [4], [3], [5], [6] and [1] to these new characteristics, we obtain spectra that maintain all the topological properties of the related ones (included compactness under some additional conditions), and this is precisely what I am going to present today. In the case when F and J are τ-homogeneous operators, these modified spectra say more on the properties of F then the previous spectra, as it will be shown in the second part of this talk by J. Appell.

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Spectral theory for homogeneous operators: part II. Applications

Jürgen Appell (Würzburg)

This is a continuations of the previous talk by Elena Giorgieri on nonlinear spectral theory for homogenous operators. The following table gives a general comparison of the three spectra introduced in Elena's talk.

Author	*Spectrum*	*Point spectrum*	*Character*
Furi-Martelli-Vignoli [9]	FMV-spectrum $\sigma_{FMV}(F,J)$	asymptotic eigenvalues $\sigma_q(F,J)$	asymptotic $(\|\|u\|\| \to \infty)$
Feng [7]	Feng spectrum $\sigma_F(F,J)$	classical eigenvalues $\sigma_p(F,J)$	global $(u \in X)$
Väth [13]	phantom $\phi(F,J)$	connected eigenvalues $\phi_p(F,J)$	local $(u \in \overline{\Omega})$

As one could expect, there are some relations between all these spectra and point spectra. For example, the Väth phantom $\phi(F,J)$ is always contained in the Furi-Martelli-Vignoli spectrum $\sigma_{FMV}(F,J)$, which in turn is contained in the Feng spectrum $\sigma_F(F,J)$. Moreover, the point phantom $\phi_p(F,J)$ is contained im the asymptotic point spectrum $\sigma_q(F,J)$. So for general operators $F,J : X \to Y$ we get the following relations.

$\phi(F,J)$	$\subseteq$	$\sigma_{FMV}(F,J)$	$\subseteq$	$\sigma_F(F,J)$
$\cup\!\mid$		$\cup\!\mid$		$\cup\!\mid$
$\phi_p(F,J)$	$\subseteq$	$\sigma_q(F,J)$		$\sigma_p(F,J)$

In the linear case $L \in \mathfrak{L}(X)$ (and $J = I$) this table essentially simplifies. Here all the spectra in the first row coincide with the usual spectrum $\sigma(L)$, and both the point spectrum $\sigma_p(L, I)$ and point phantom $\phi_p(L, I)$ coincide with the usual point spectrum $\sigma_p(L)$.

Even if one restricts the class of nonlinear operators in consideration, the above table may simplify. We confine ourselves to the case of *τ-homogeneous operators* F and J, i.e.

$$F(tu) = t^\tau F(u), \quad J(tu) = t^\tau J(u) \qquad (t > 0,\, u \in X). \tag{29}$$

The following two theorems have been proved in [2].

Theorem 4 (Coincidence theorem). *Let X and Y be infinite dimensional Banach spaces, and suppose that $F, J : X \to Y$ satisfy (29) for some $\tau > 0$. Then*

$$\sigma_{FMV}(F,J) = \sigma_F(F,J) = \phi(F,J), \quad \sigma_q(F,J) \supseteq \sigma_p(F,J) = \phi_p(F,J).$$

Theorem 5 (Discreteness theorem). *Let X and Y be infinite dimensional Banach spaces, and suppose that $F, J : X \to Y$ are odd, $[F]_A = 0$ (i.e., F is compact), and $[J]_a > 0$. Then*

$$\sigma_{FMV}(F,J) \setminus \{0\} \subseteq \sigma_q(F,J), \quad \sigma_F(F,J) \setminus \{0\} \subseteq \sigma_p(F,J),$$

and

$$\phi(F,J) \setminus \{0\} \subseteq \phi_p(F,J).$$

Theorem 5 shows that, for F compact and odd, and J "sufficiently regular" and odd, each nonzero spectral value is actually an eigenvalue (in a sense to be made precise). For F compact and linear and $J = I$ this is a classical fact.

To illustrate how these theorems apply to nonlinear problems, we consider the *eigenvalue problem for the p-Laplacian* which consists in finding solutions $u \not\equiv 0$ of

$$\begin{cases} -\mathrm{div}\,(|\nabla u|^{p-2}\,\nabla u)(x) = \mu\,|u(x)|^{p-2}\,u(x) & \text{in } G \\ u(x) \equiv 0 & \text{on } \partial G \end{cases} \tag{30}$$

where $G \subset \mathbb{R}^n$ is a bounded domain. Although this problem makes sense for $1 < p < \infty$, we restrict ourselves to the case $2 \le p < \infty$. The problem (30) may be reformulated as equivalent operator equation in weak form

$$F_p(u) = \lambda J_p(u), \tag{31}$$

where $\lambda = 1/\mu$, and $F_p, J_p : W_0^{1,p}(G) \to W^{-1,p'}(G)$ $(p' = p/(p-1))$ are defined by $F_p(u) = |u|^{p-2}\, u$ and

$$\langle J_p(u), v\rangle = -\int_G (|\nabla u(x)|^{p-2}\, \nabla u(x), \nabla v(x))\, dx \quad (u, v \in W_0^{1,p}(G)),$$

respectively. Equation (31) has been studied by many authors, e.g. by Drábek et al. in [3–6]. Interestingly, the eigenvalue theory for the problem (30) has many features in common with the classical *linear* eigenvalue problem $-\Delta u(x) = \mu u(x)$, which is a special case of (30) for $p = 2$. For instance, the first eigenvalue μ_1 of (30) is always positive and simple and may be “calculated” as Rayleigh quotient

$$\mu_1 = \inf_{\substack{u \in W_0^{1,p}(G) \\ u \neq 0}} \frac{\displaystyle\int_G |\nabla u(x)|^p \; dx}{\displaystyle\int_G |u(x)|^p \; dx}.$$

Moreover, the corresponding eigenfunction $u_1 \in W_0^{1,p}(G)$ is positive on G and simple (in the sense that any other eigenfunction is a scalar multiple of u_1). This function has the same “variational characterization” as in the linear case $p = 2$: it minimizes the functional $\Psi_p : W_0^{1,p}(G) \to \mathbb{R}$ defined by $\Psi_p(u) = \frac{1}{p}\langle J_p(u), u\rangle$, subject to the constraint

$$\frac{1}{p}\int_G |u(x)|^{p-2}\, u(x)\, dx = 1.$$

Finally, we point out that there is a famous so-called *nonlinear Fredholm alternative* (see [8, 11, 12]) which implies that the operator $J_p - \mu F_p = \mu(\lambda J_p - F_p)$ is *onto* for $\mu < \mu_1$, while it is *not onto* for $\mu = \mu_1$. However, the coincidence and discreteness theorems given above allow us a more precise statement. The following is just a reformulation of Theorems 4 and 5.

Theorem 6 (Nonlinear Fredholm alternative). *Suppose that $J : X \to Y$ is an odd τ-homogeneous homeomorphism with $[J]_a > 0$, and $F : X \to Y$ is odd, τ-homogeneous and compact. Let $\lambda \neq 0$. Then the following four assertions are equivalent.*

(a) *The eigenvalue problem* (30) *has only the trivial solution $u = 0$.*
(b) *The operator $\lambda J - F$ is stably solvable, $[\lambda J - F]_a > 0$, and $[\lambda J - F]_q > 0$.*
(c) *The operator $\lambda J - F$ is epi on each $\Omega \in \mathfrak{O}(X)$, $[\lambda J - F]_a > 0$, and $[\lambda J - F]_b > 0$.*
(d) *The operator $\lambda J - F$ is strictly epi on some $\Omega \in \mathfrak{O}(X)$, and*

$$\inf\, \{\|\lambda J(u) - F(u)\| : u \in \partial\Omega\} > 0.$$

We claim that the operators F_p and J_p satisfy the hypotheses of Theorem 6 in the spaces $X = W_0^{1,p}(G)$ and $Y = X^* = W^{-1,p'}(G)$. In fact, since $J_p : X \to Y$ is continuous, strictly monotone, coercive (it is here that we use the restriction $p \geq 2!$), odd, and $(p-1)$-homogeneous, it is an *isomorphism*, by Minty's celebrated theorem [10]. Moreover, the coercivity also implies that $[J_p]_a > 0$. Finally, the operator $F_p : X \to Y$ is continuous, compact (by Krasnosel'skij's theorem and the compactness of the imbedding $X \hookrightarrow L_p(G)$), odd, and also $(p-1)$-homogeneous. So Theorem 6 implies that, whenever μ is not a classical eigenvalue of (2), then the operator $J_p - \mu F_p$ is not only onto, but even stably solvable and strictly epi. This makes it possible to obtain existence, uniqueness, and stability results for nonlinear perturbations of (31).

Several other applications of nonlinear spectra may be found in Chapter 12 of the recent monograph [1].

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Numerical Ranges for Nonlinear Operators: A Survey

Jürgen Appell (Würzburg)

This talk was supposed to be given by E. De Pascale (Cosenza, Italy) who was unable to come to Oberwolfach.

The purpose of the talk is to give an overview of the definition and properties of numerical ranges for both linear and nonlinear operators in Hilbert or Banach spaces. For linear operators in Hilbert spaces this goes back to Toeplitz [12], for linear operators in Banach spaces to Bauer [1], and, independently, to Lumer [7]. In the nonlinear case, corresponding definitions have been given in the Hilbert space setting by Zarantonello [13–15], and in the Banach space setting by Rhodius [9–11], Martin [8], Dörfner [4], and Feng [5]. Somewhat different notions of numerical ranges are due to Furi, Martelli and Vignoli [6], Bonsall, Cain and Schneider [2], and Canavati [3].

Numerical ranges and radii have applications in matrix theory, numerical analysis, approximation theory, functional analysis, operator theory, system theory, and even in quantum mechanics. There are also useful for "localizing" the spectrum of an operator in the complex plane. This provides the connection with the topics dealt with in the Miniworkshop.

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