Mathematisches Forschungsinstitut
Oberwolfach

Report No. 1/2004

Combinatorics

Organised by
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January 4th – January 10th, 2004

Introduction by the Organisers

The conference was organized by László Lovász (Redmond) and Hans Jürgen Prömel (Berlin). The programme consisted of 15 lectures, supplemented by 21 shorter contributions, and covered many areas in Combinatorics such as partition theory, discrete geometry, homomorphisms and lattices, extremal combinatorics, graph theory, random structures, and additive number theory. The aim of the workshop was to emphasize the underlying methods that are common to many of these combinatorial branches and that act as both driving forces and organizing principles of the field. The diversity of the topics and participants stimulated a lot of fruitful discussion between the different branches and gave rise to new collaborations, in particular for the younger generation of researchers.

In total, 51 scientists participated in this meeting; almost 40 came from countries other than Germany. The organizers and participants thank the *Mathematisches Forschungsinstitut Oberwolfach* for providing an inspiring setting for this conference. In the following we include the abstracts in alphabetical order.

Workshop on Combinatorics

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Abstracts

CutNorm, Grothendieck's Inequality, and Approximation Algorithms for Dense Graphs

Noga Alon
(joint work with Assaf Naor)

The *cut-norm* $||A||_C$ of a real matrix $A = (a_{ij})_{i\in R, j\in S}$ with a set of rows indexed by R and a set of columns indexed by S is the maximum, over all $I \subset R,\ J \subset S$, of the quantity $|\sum_{i\in I, j\in J} a_{ij}|$. This concept plays a major role in the work of Frieze and Kannan on efficient approximation algorithms for dense graph and matrix problems, [3] (see also [1] and its references). Although the techniques in [3] enable the authors to approximate efficiently the cut-norm of an n by m matrix with entries in $[-1, 1]$ up to an **additive** error of ϵnm, there is no known polynomial algorithm that approximates the cut-norm of a general real matrix up to a constant multiplicative factor.

Let CUT NORM denote the computational problem of computing the cut-norm of a given real matrix. Here we first observe that the CUT NORM problem is MAX SNP hard, and then provide an efficient approximation algorithm for the problem. This algorithm finds, for a given matrix $A = (a_{ij})_{i\in R, j\in S}$, two subsets $I \subset R$ and $J \subset S$, such that $|\sum_{i\in I, j\in J} a_{ij}| \geq \rho||A||_C$, where $\rho > 0$ is an absolute constant. We first describe a deterministic algorithm that supplies a rather poor value of ρ, and then describe a randomized algorithm that provides a solution of expected value greater than 0.56 times the optimum.

The algorithm combines semidefinite programming with a novel rounding technique based on (the proofs of) Grothendieck's Inequality. This inequality, first proved in [6], is a fundamental tool in Functional Analysis, and has several interesting applications in this area. We will actually use the matrix version of Grothendieck's inequality, formulated in [10]. In order to apply semidefinite programming for studying the cut-norm of an n by m matrix $A = (a_{ij})$, it is convenient to first study another norm,

$$||A||_{\infty\mapsto 1} = \max \sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}x_i y_j,$$

where the maximum is taken over all $x_i, y_j \in \{-1, 1\}$.

It is not difficult to show, that for every matrix A,

$$4||A||_C \geq ||A||_{\infty\mapsto 1} \geq ||A||_C,$$

and hence a constant approximation of any of these norms provides a constant approximation of the other.

The value of $||A||_{\infty\mapsto 1}$ is given by the following quadratic integer program

$$\text{Maximize} \quad \sum_{ij} a_{ij} x_i y_j \tag{1}$$

$$\text{subject to} \quad x_i, y_j \in \{-1, 1\} \quad \text{for all} \quad i, j.$$

The obvious semidefinite relaxation of this program is

$$\text{Maximize} \quad \sum_{ij} a_{ij} u_i \cdot v_j \tag{2}$$

$$\text{subject to} \quad ||u_i|| = ||v_j|| = 1,$$

where here $u_i \cdot v_j$ denotes the inner product of u_i and v_j, which are now vectors of (Euclidean) norm 1 that lie in an arbitrary Hilbert space. Clearly we may assume, without loss of generality, that they lie in an $n+m$-dimensional space.

This semidefinite program can be solved, using well known techniques (see [5]) within an additive error of ϵ, in polynomial time (in the length of the input and in the logarithm of $1/\epsilon$.) The main problem is the task of rounding this solution into an integral one. A first possible attempt is to imitate the technique of Goemans and Williamson in [7], that is, given a solution u_i, v_j to the above program, pick a random vector z and define $x_i = \text{sign}(u_i \cdot z)$ and $y_j = \text{sign}(v_j \cdot z)$. It is easy to check that the expected value of $x_i y_j$ satisfies $E(x_i y_j) = \frac{2}{\pi} \arcsin(u_i \cdot v_j)$, and as $\arcsin(t)$ and t differ only in constant factors for all $-1 \leq t \leq 1$, one could hope that this will provide an integral solution whose value is at least some absolute constant fraction of the value of the optimal solution. This reasoning is, unfortunately, incorrect, as some of the entries a_{ij} may be positive and some may be negative, (in fact, the problem is interesting only if this is the case, since otherwise either $x_i = y_j = 1$ or $x_i = -y_j = 1$ for all i, j supplies the required maximum). Therefore, even if each single term $a_{ij} u_i \cdot v_j$ is approximated well by its integral rounding $a_{ij} x_i y_j$, there is no reason to expect the sum to be well-approximated, due to cancellations. We thus have to compare the value of the rounded solution to that of the semidefinite program on a global basis. Nesterov [11] obtained a result of this form for the problem of approximating the maximum value of a quadratic form $\sum_{ij} b_{ij} x_i x_j$, where $x_i \in \{-1, 1\}$, but only for the special case in which the matrix $B = (b_{ij})$ is positive semidefinite. While his estimate is global, his rounding is the same simple rounding technique of [7] described above. As explained before, some new ideas are required in our case in order to get any nontrivial result.

Luckily, there is a well known inequality of Grothendieck, which asserts that the value of the semidefinite program (2) and that of the integer program (1) can differ only by a constant factor. The precise value of this constant, called

Grothendieck's constant and denoted by K_G, is not known, but it is known that its value is at most $\frac{\pi}{2\ln(1+\sqrt{2})} = 1.782...$ ([8]) and at least $\frac{\pi}{2} = 1.570...$ ([6]). Stated in other words, the integrability gap of the problem is at most K_G. (Krivine mentions in [8] that he can improve the lower bound, but such an improvement has never been published).

It follows that the value of the semidefinite program (2) provides an approximation of $||A||_{\infty\mapsto 1}$ up to a constant factor. This, however, still does not tell us how to round the solution of the semidefinite program into an integral one with a comparable value. Indeed, this task requires more work, and is carried out in the full paper a preliminary version of which will appear in the proceedings of STOC 2004.

We describe three rounding techniques. The first one is a deterministic procedure, which combines Grothendieck's Inequality with some facts about four-wise independent random variables, in a manner that resembles the technique used in [2] to approximate the second frequency moment of a stream of data under severe space constraints. The second rounding method is based on Rietz' proof of Grothendieck's Inequality [12]. This proof supplies a better approximation guarantee for the special case of positive semidefinite matrices A, where the integrality gap can be shown to be precisely $\pi/2$, and implies that Nesterov's analysis for the problem he considers in [11] is tight.

The third technique, which supplies the best approximation guarantee, is based on Krivine's proof of Grothendieck's Inequality. Here we use the vectors u_i, v_j which form a solution of the semidefinite program (2) to construct some other unit vectors u'_i, v'_j, which are first shown to exist in an infinite dimensional Hilbert space, and are then found, using another instance of semidefinite programming, in an $n+m$-dimensional space. These vectors can now be rounded to $\{-1, 1\}$ in order to provide an integral solution for the original problem (1) in a rather simple way. We note that there are several known techniques for modifying the solution of a semidefinite program before rounding it, see [13], [9], [4]. Here, however, the modification seems more substantial.

We believe that our techniques will have further applications, as they provide a method for handling problems in which there is a possible cancellation between positive and negative terms. It seems that there are additional interesting problems of this type. Moreover, unlike the semidefinite based approximation algorithms for MAX CUT, MAX 2SAT and related problems, suggested in the seminal paper of [7] and further developed in many subsequent papers, the problem considered here has no known constant approximation algorithm, and the semidefinite programming and its rounding appear to be essential in order to obtain any constant approximation guarantee, and not only in order to improve the constants ensured by appropriate combinatorial algorithms.

References

[1] N. Alon, W. F. de la Vega, R. Kannan and M. Karpinski, Random sampling and approximation of MAX-CSP problems, Proc. of the 34^{th} ACM STOC, ACM Press (2002), 232–239.

[2] N. Alon, Y. Matias and M. Szegedy, The space complexity of approximating the frequency moments, Proc. of the 28^{th} ACM STOC, ACM Press (1996), 20-29. Also; J. Comp. Sys. Sci. 58 (1999), 137-147.

[3] A. M. Frieze and R. Kannan, Quick Approximation to matrices and applications, Combinatorica **19** (2) (1999), 175–200.

[4] U. Feige and M. Langberg, The RPR^2 Rounding Technique for Semidefinite Programs, Proceedings of the 28th International Colloquium on Automata, Languages and Programming, Crete, Greece, 2001, 213–224.

[5] M. Grötschel, L. Lovász and A. Schrijver, The ellipsoid method and its consequences in combinatorial optimization, Combinatorica 1 (1981), 169–197.

[6] A. Grothendieck, Résumé de la théorie métrique des produits tensoriels topologiques, Bol. Soc. Mat. Sao Paulo 8 (1953), 1–79.

[7] M. X. Goemans and D. P. Williamson, Improved approximation algorithms for maximum cut and satisfiability problems using semidefinite programming, J. ACM 42 (1995), 1115–1145.

[8] J. L. Krivine, Sur la constante de Grothendieck, C. R. Acad. Sci. Paris Ser. A-B 284 (1977), 445–446.

[9] M. Lewin, D. Livnat and U. Zwick, Improved rounding techniques for the MAX 2-SAT and MAX DI-CUT problems, Proceedings of the 9th Conference on Integer Programming and Combinatorial Optimization, Cambridge, Massachusetts, 2002, 67–82.

[10] J. Lindenstrauss and A. Pełczyński, Absolutely summing operators in L_p-spaces and their applications, Studia Math. 29 (1968), 275-326.

[11] Y. E. Nesterov, Semidefinite relaxation and nonconvex quadratic optimization, Optimization Methods and Software 9 (1998), 141–160.

[12] R. E. Rietz, A proof of the Grothendieck inequality, Israel J. Math. 19 (1974), 271–276.

[13] U. Zwick, Outward rotations: a tool for rounding solutions of semidefinite programming relaxations, with applications to MAX CUT and other problems, Proceedings of the 31th Annual ACM Symposium on Theory of Computing (STOC), Atlanta, Georgia, 1999, 679–687.

Blockers, Ideals and some Turán-type Questions

Anders Björner

(joint work with Axel Hultman, Irena Peeva and Jessica Sidman [1, 2])

The point of departure are the theorems of Li & Li and Kleitman & Lovász (from 1981) describing generators for certain ideals, see [3]. The immediate combinatorial interest of these theorems is that they in a useful way describe ideals with the property that (upper) bounded independence number and (lower) bounded chromatic number of a given graph are equivalent to membership of the corresponding graph polynomial in these ideals. But the theorems are also interesting from a ring-theoretic point of view, since they suggest a combinatorial procedure for constructing generators for vanishing ideals of subspace arrangements.

The work presented was:

(1) The *blocker construction* $A \mapsto A^*$ for antichains in finite posets, generalizing the well-known concept in Boolean lattices (set clutters). Particularly how to compute blockers for symmetric antichains in the partition lattice Π_n. This procedure involves both the refinement order and the dominance order on the set of all number partitions of n.

(2) The construction of the *blocker ideal* $B_{\mathcal{A},\mathcal{H}}$ for a subspace arrangement $\mathcal{A}$ embedded in a hyperplane arrangement $\mathcal{H}$. This ideal is contained in the vanishing ideal $I_{\mathcal{A}}$ for the union of the subspaces in $\mathcal{A}$, and

$$B_{\mathcal{A},\mathcal{H}} = I_{\mathcal{A}} \quad \Rightarrow \quad \mathcal{A}^{**} = \mathcal{A},$$

where $\mathcal{A}^*$ denotes the blocker of $\mathcal{A}$ w.r.t. the intersection lattice of $\mathcal{H}$.

(3) The fact that $B_{\mathcal{A},\mathcal{H}} = I_{\mathcal{A}}$ implies that a minimal blocking set for $\mathcal{A}$ has size equal to the minimal size of a flat in the blocker $\mathcal{A}^*$. Some extremal results (e.g. Turán's theorem) can be deduced this way.

References

[1] A. Björner and A. Hultman, *A note on blockers in posets*, preprint, 2004. (http://arXiv.org/abs/math.CO/0403094)

[2] A. Björner, I. Peeva and J. Sidman, *Subspace arrangements defined by products of linear forms*, preprint, 2003. (http://arXiv.org/abs/math.CO/0401373)

[3] László Lovász, *Stable sets and polynomials*, Discrete Math. **124** (1994), 137–153.

Random Geometric Graphs

Béla Bollobás
(joint work with Paul Balister, Amites Sarkar and Mark Walters)

Random geometric graphs were introduced by Gilbert [6] in 1961, and in the past forty years many variants of them have been studied in great detail (see Meester and Roy [7], Penrose [8]). The aim of the talk is to present a number of recent results obtained jointly with Paul Balister, Amites Sarkar and Mark Walters on a variety of geometric random graphs.

Gilbert's *disc model* G_r is defined as follows. Place points $\{x_i\}$ in $\mathbb{R}^2$ according to a Poisson process with intensity 1 and let G_r be the random graph with vertex set $\{x_i\}$ and edges $x_i x_j$ whenever $|x_i - x_j| \le r$. Equivalently, let D_r be the disc of radius r with centre the origin, and join each x_i to every x_j in the disc $x_i + D_r$ of radius r centred at x_i. There is a critical area a_c such that if $|D_r| = \pi r^2 < a_c$ then a.s. G_r has no infinite component (G_r does not percolate), while if $|D_r| > a_c$ then G_r percolates a.s. The proven bounds on a_c are still rather weak, with almost a factor 5 between the upper and lower bounds. In the talk we present the result due to Balister, Bollobás and Walters [4] that $4.508 < a_c < 4.515$ *with probability* 99.99%. (The probability is due to the uncertainty of numerically evaluating a large integral.) For the critical area s_c of a square rather than a disc, defined analogously, the corresponding bounds are $4.392 < s_c < 4.398$.

Problems concerning *ad hoc* networks of radio transceivers inspire the following considerable extension of the disc model. Place points $\{x_i\}$ in $\mathbb{R}^d$ according to a Poisson process with intensity 1. Then, independently for each x_i, choose a bounded region A_{x_i} from some fixed distribution and let $\mathcal{G}$ be the random directed graph with vertex set $\{x_i\}$ and edges $\vec{x_i x_j}$ whenever $x_j \in x_i + A_{x_i}$. The main result of Balister, Bollobás and Walters [3] states that for any $\eta > 0$, if the regions $x_i + A_{x_i}$ do not overlap too much (i.e., satisfy a somewhat technical precise condition), then $\mathcal{G}$ has an infinite directed path provided the expectation of the area $|A_{x_i}|$ of the domain A_{x_i} is at least $1 + \eta$. (It is trivial that the area has to be at least 1.) One example where these conditions hold, and we obtain percolation, is in dimension d

with A_{x_i} a ball of volume $1+\eta$, where η tends to zero as d tends to infinity. Another example is in two dimensions, where the A_{x_i} are randomly oriented sectors of a disk of angle $2\pi\varepsilon$ and area $1+\eta$. In this case we can let η tend to zero as ε tends to zero. Yet another special case of this theorem is the result proved independently in [2] and by Franceschetti et al [5] that, given $\eta > 0$, if $\varepsilon > 0$ is small enough, in $\mathbb{R}^2$ we may take each A_{x_i} to be a 'thin' annulus $A = \{x \in \mathbb{R}^2 : \ r(1-\varepsilon) \leq |x| \leq r\}$ of area $1 + \eta$.

In the talk we shall examine some *finite* geometric random graphs as well. Let $\mathcal{P}$ be a Poisson process of intensity one in a square S_n of area n. We construct a random geometric graph $G_{n,k}$ by joining each point of $\mathcal{P}$ to its k nearest neighbors. Recently, Xue and Kumar [9] proved that if $k = 0.074 \log n$ then the probability that $G_{n,k}$ is connected tends to zero as $n \to \infty$, while if $k = 5.1774 \log n$ then the probability that $G_{n,k}$ is connected tends to one as $n \to \infty$. They conjectured that the threshold for connectivity is $k = \log n$. Recently, Balister, Bollobás, Sarkar and Walters [1] have improved these lower and upper bounds to $k = 0.3043 \log n$ and $k = 0.5139 \log n$, respectively, disproving this conjecture, and have proved reasonably good bounds for some generalizations of this problem.

References

[1] P. Balister, B. Bollobás, A. Sarkar and M. Walters, Connectivity of random geometric graphs, to appear

[2] P. Balister, B. Bollobás and M. Walters, Continuum percolation with steps in an annulus, *Advances of Applied Probability,* to appear

[3] P. Balister, B. Bollobás and M. Walters, Random transceiver networks, to appear

[4] P. Balister, B. Bollobás and M. Walters, Continuum percolation in the square and the disc, to appear

[5] M. Franceschetti, L. Booth, M. Cook, R. Meester and J. Bruck, *IEEE Trans. on Information Theory,* to appear

[6] E.N. Gilbert, Random plane networks, *Journal of the Society for Industrial Applied Mathematics* **9** (1961), 533-543.

[7] R. Meester and R. Roy, *Continuum Percolation,* Cambridge University Press, 1996, x+238 pp.

[8] M.D. Penrose, *Random Geometric Graphs,* Oxford University Press, 2003, xiii+330 pp.

[9] F. Xue and P.R. Kumar, The number of neighbors needed for connectivity of wireless networks, *Wireless Networks*, to appear

The Number of Linear Extensions of the Boolean Lattice

Graham Brightwell
(joint work with Prasad Tetali [1])

Let $L(P)$ denote the number of linear extensions of a poset P. A natural problem is to estimate $L(P)$ when P is the Boolean lattice Q^t, consisting of the subsets of $\{1, 2, \ldots, t\}$, ordered by inclusion. This problem was apparently first posed by Richard Stanley, although it has also been raised by several others independently.

A trivial lower bound on $L(Q^t)$ is $\prod_{j=0}^{t} \binom{t}{j}!$, and a simple upper bound is $\binom{t}{\lfloor t/2 \rfloor}^{2^t}$; these bounds can be written as

$$\log \binom{t}{\lfloor t/2 \rfloor} - \frac{3}{2} \log e + o(1) \leq \frac{\log(L(Q^t))}{2^t} \leq \log \binom{t}{\lfloor t/2 \rfloor}.$$

(All logarithms are base 2.)

The only previous improvement on these trivial bounds was made by Sha and Kleitman [4], who improved the upper bound to

$$L(Q^t) \leq \prod_{j=0}^{t} \binom{t}{j}^{\binom{t}{j}} \leq \prod_{j=0}^{t} \binom{t}{j}! \exp(2^t),$$

yielding

$$\frac{\log(L(Q^t))}{2^t} \leq \log \binom{t}{\lfloor t/2 \rfloor} - \frac{1}{2} \log e + o(1).$$

In fact, the Sha-Kleitman bound can be generalised to any ranked poset satisfying the LYM condition (see [1]).

We prove the following result, which shows that (as was generally expected) the trivial lower bound gives the correct constant term in the asymptotic expansion:

$$\frac{\log(L(Q^t))}{2^t} = \log \binom{t}{\lfloor t/2 \rfloor} - \frac{3}{2} \log e + O\left(\frac{\ln t}{t}\right).$$

Our proof is based on what seems to be emerging as an "entropy method" developed by Jeff Kahn [2], and used by him [3] to give a short and natural proof of the Kleitman-Markowsky bound for Dedekind's problem concerning the number of antichains in the Boolean lattice.

In the case where the poset P is bipartite, a small adaptation of Kahn's proof from [2] yields an extremal result. For P a bipartite poset on n elements, with two ranks A and B, such that every element of A is below exactly u elements of B, and every element of B is above exactly d elements of A, we have

$$L(P) \leq n! \binom{d+u}{u}^{-n/(d+u)}.$$

This result is best possible for n a multiple of $d+u$.

References

[1] G.R. Brightwell and P. Tetali, The number of linear extensions of the Boolean lattice, to appear in *Order*.

[2] J. Kahn, An entropy approach to the hard-core model on bipartite graphs, *Combin. Prob. Comp.* **10** (2001), 219–237.

[3] J. Kahn, Entropy, independent sets and antichains: a new approach to Dedekind's problem, *Proc. Amer. Math. Soc.* **130** (2002), 371–378.

[4] J. Sha and D. J. Kleitman, The number of linear extensions of subset ordering, *Discrete Math.*, **63** (1987), 271–279.

Coloring Claw-free Graphs

Maria Chudnovsky

(joint work with Paul Seymour)

A graph is called *claw-free* if it has no induced subgraph isomorphic to $K_{1,3}$. Line graphs are a well-known class of claw-free graphs, but there are others, such as *circular interval graphs* and subgraphs of the *Schläfli graph* (a circular interval graph is obtained from a collection of circular intervals and points on a circle by making two points adjacent if they belong to the same interval). Recently we were able to prove that all claw-free graphs in which every vertex is in a stable set of size three, can be built from the classes mentioned above, together with some others, by combining them in prescribed ways (this work is described in another paper in this issue).

Claw-free graphs being a generalization of line graphs, it is natural to ask what properties of line graphs can be extended to all claw-free graphs. Vizing's theorem [1] gives a bound on the chromatic number, χ, of a line graph, in terms of the size of a maximum clique, ω, namely $\chi \leq \omega + 1$. Is there a similar bound for all claw-free graphs? Does there exist a function f such that if G is a claw-free graph then $\chi(G) \leq f(\omega(G))$? It is easy to see that such f exists, in fact $\chi(G) \leq \omega(G)^2$ (the neighborhood of a vertex in a clique of size ω is the union of at most ω cliques).

One might hope to get closer to Vizing's bound, asking whether f is a linear function. Unfortunately the answer to this question is negative. If G is the complement of a triangle free graph, then $\chi(G) \geq \frac{|V(G)|}{2}$, and yet $\omega(G)$ may be of order $\sqrt{(|V(G)|)}$. However, if we insist that G contains a stable set of size three, and is connected (to prevent taking disjoin union with large complement triangle-free graphs), then a much stronger result is true. We prove:

Theorem 1 *Let G be a connected, claw-free graph and assume that G contains a stable set of size three. Then $\chi(G) \leq 2\omega(G)$.*

This bound is best possible. The proof of 1 uses the structure theorem mentioned above: first we verify the result for the basic classes of claw-free graphs, and then prove that it is preserved under the operations. This proves the theorem for those claw-free graphs that satisfy the hypotheses of the structure theorem, namely claw-free graphs where every vertex is in a stable set of size three. But it turns out that having proved the result for the part of the graph where every vertex is in a stable set of size three, one can always figure out the "important" information about vertices not in stable sets of size three, and finish the proof.

There is a slightly worse, but still linear bound on χ in terms of ω, that has a short proof, without using the structure theorem, and we include it here.

Theorem 2 *Let G be a connected, claw-free graph and assume that G contains a stable set of size three. Then $\chi(G) \leq 4\omega(G)$.*

In fact, we prove the following stronger statement that clearly implies 2. This was conjectured by N. Linial during the Oberwolfach meeting.

Theorem 3 *Let G be a connected, claw-free graph and assume that G contains a stable set of size three. Then every vertex of G has degree at most $4\omega(G)$.*

Proof. We use induction on $|V(G)|$. Let v be a vertex of maximum degree in G and let N be the set of neighbors of v. Since G is claw-free and contains a stable set of size three, $V(G) \neq N \cup \{v\}$ and there exists a vertex $u \in V(G) \setminus (N \cup \{v\})$ such that the graph $G \setminus u$ is connected. We may assume $G \setminus u$ does not contain a stable set of size three, for otherwise the result follows inductively. Let A be the set of neighbors of u in G and B the set of non-neighbors. Since G contains a stable set of size three, and $G \setminus u$ does not, it follows that there exist two non-adjacent vertices b_1, b_2 in B. Since G is connected, A is non-empty. For $i = 1, 2$ let N_{b_i} be the set of neighbors of b_i in A. Since every vertex in $N_{b_1} \cap N_{b_2}$ would be the center of a claw in G, $N_{b_1} \cap N_{b_2} = \emptyset$. Since $G \setminus u$ contains no stable set of size three, $A \setminus N_{b_i}$ is a clique for $i = 1, 2$, and A is the union of two cliques. Also since $G \setminus u$ contains no stable set of size three, $N_{b_1} \cup N_{b_2} = A$. So for every pair of non-adjacent vertices in B, the sets of their neighbors in A partition A. It follows that $G|B$ does not contain the complement of an odd cycle, and so $G|B$ is the complement of a bipartite graph, in particular B is the union of two cliques. But now G is the union of four cliques, so $\omega \geq \frac{|V(G)|}{4}$, and the theorem holds. This proves 3. ▮

References

[1] D.B. West, Introduction to Graph Theory, (Prentice Hall, 2001),

The Homology of a Locally Finite Graph with Ends

Reinhard Diestel

When one studies the homology aspects of an infinite graph – in graph-theoretic language, the properties of its cycle space – one can observe a curious phenomenon: while all the basic properties of the cycle space of a finite graph remain true (and trivial) also for infinite graphs, few of the less trivial theorems carry over.

Surprisingly, the situation can be remedied simultaneously for all those theorems that fail in the infinite case by using a different homology for locally finite graphs: not the simplicial homology of the graph itself, but a variant of the singular homology of its Freudenthal compactification.

Our approach permits the extension to locally finite infinite graphs of the following finite theorems, whose infinite analogues all fail with the usual simplicial homology:

- Tutte's theorem that the peripheral (ie., non-separating and induced) cycles of a 3-connected graph generate its cycle space;

- Whitney's theorem that a graph has a combinatorial dual if and only if it is planar;

- Euler's theorem that a connected graph admits an Euler tour iff its edge set lies in its cycle space (the infinite analogue of an Euler tour being a closed topological curve in the compacification that traverses every edge exactly once);
- Gallai's theorem that the vertex set of a graph can be partitioned into two sets each inducing an element of its cycle space;
- MacLane's theorem that a graph is planar iff its cycle space has a set of generators such that every edge lies in at most two of these;
- Tutte's theorem that a 3-connected graph is planar iff every edge lies on at most two peripheral cycles;
- the Tutte - Nash-Williams *tree-packing* theorem that a graph has k edge-disjoint spanning trees iff every vertex partition, into ℓ sets say, is crossed by at least $k(\ell - 1)$ edges;
- the *4-colour-theorem* (expressed dually in terms of 4-flows) that the edge set of a planar bridgeless graph is a union of two elements of its cycle space (ie., has a 4-flow).

Furthermore, the following easy facts about the cycle space of a finite graph extend to non-trivial theorems about locally finite graphs with this new cycle space:

- Every element of the cycle space is an edge-disjoint union (not just a sum) of cycles.
- A non-empty set of edges lies in the cycle space iff it meets every finite cut in an even number of edges, and it lies in the cocycle space (ie., is a cut) iff it meets every finite element of the cycle space in an even number of edges.
- The fundamental cycles of any spanning tree generate the cycle space (the generalization is based on *topological spanning trees*, path-connected subspaces containing all the vertices and ends but no continuous 1–1 image of S^1; note that these 'trees' need not induce connected subgraphs, as their path-connectedness can result from topological paths including ends).
- A set of edges lies in the cycle space iff in the subgraph it induces all vertex degrees are even.

The generalization of the last statement involves the definition of 'degrees' also for ends. An end has *degree k* if there are k but not $k + 1$ edge-disjoint infinite paths converging to it. If there is no such k, it has *infinite degree*. Infinite end degrees are also classified into 'odd' and 'even' in a more complicated way, which however is essential for the generalization of the above statement.

The new notion of end degrees motivated by these results seems to open up new possibilities for an 'extremal' branch of infinite graph theory. For example, is there a function $f\colon \mathbb{N} \to \mathbb{N}$ such that every locally finite graph whose vertices and ends all have degree at least $f(k)$ contains a k-connected subgraph? (Note that since infinite trees can have large minimum degree, vertex degrees alone do not force any dense substructures.)

Another natural area of application lies in Hamiltonicity problems. Define a *Hamilton circle* in a graph G as a homeomorphic image of S^1 in its Freudenthal compactification that contains all its vertices. Does every 4-connected planar locally finite graph have a Hamilton circle (extending Tutte's theorem)? Does the square of every 2-connected locally finite graph have a Hamilton circle (extending Fleischner's theorem)?

See [6] for an introductory overview of these results and numerous further problems.

References

[1] H. Bruhn, The cycle space of a 3-connected locally finite graph is generated by its finite and infinite peripheral circuits, *J. Combin. Theory B* (to appear).

[2] H. Bruhn and R. Diestel, Duality in infinite graphs, in preparation.

[3] H. Bruhn, R. Diestel and M. Stein, Cycle-cocycle partitions and faithful cycle covers for locally finite graphs, preprint 2003.

[4] H. Bruhn and M. Stein, MacLane's planarity criterion for locally finite graphs, preprint 2003.

[5] H. Bruhn and M. Stein, Characterizing the cycle space of a locally finite graph by vertex and end degrees, in preparation.

[6] R. Diestel, The cycle space of an infinite graph, *Comb. Probab. Computing* (to appear).
`http://www.math.uni-hamburg.de/home/diestel/papers/CyclesExpository.pdf`

[7] R. Diestel and D. Kühn, On infinite cycles I, *Combinatorica* (to appear).

[8] R. Diestel and D. Kühn, On infinite cycles II, *Combinatorica*, (to appear).

[9] R. Diestel and D. Kühn, Topological paths, cycles and spanning trees in infinite graphs, *Europ. J. Combinatorics* (to appear).

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Graph Products, Fourier Analysis and Spectral Techniques

Ehud Friedgut
(joint work with Noga Alon, Irit Dinur and Benny Sudakov)

We consider powers of regular graphs defined by the weak graph product and give a characterization of maximum-size independent sets for a wide family of base graphs which includes, among others, complete graphs, line graphs of regular graphs which contain a perfect matching and Kneser graphs. In many cases this also characterizes the optimal colorings of these products.

We show that the independent sets induced by the base graph are the only maximum-size independent sets. Furthermore we give a qualitative stability statement: any independent set of size close to the maximum is close to some independent set of maximum size.

Our approach is based on Fourier analysis on Abelian groups and on Spectral Techniques. To this end we develop some basic lemmas regarding the Fourier transform of functions on $\{0 \dots r-1\}^n$, generalizing some useful results from the $\{0,1\}^n$ case.

Consider the following combinatorial problem:

Assume that at a given road junction there are n three-position switches that control the red-yellow-green position of the traffic light. You are told that whenever you change the position of *all* the switches then the color of the light changes. Prove that in fact the light is controlled by only one of the switches.

The above problem is a special case of the problem we wish to tackle in this paper, characterizing the optimal colorings and maximal independent sets of products of regular graphs. The configuration space of the switches described above can be modeled by the n-fold product of K_3. Let us begin by defining the weak graph product of two graphs.

The *weak product* of G and H, denoted by $G \times H$ is defined as follows: the vertex set of $G \times H$ is the Cartesian product of the vertex sets of G and H. Two vertices (g_1, h_1) and (g_2, h_2) are adjacent in $G \times H$ if $g_1 g_2$ is an edge of G and $h_1 h_2$ is an edge of H. The "times" symbol, $\times$, is supposed to be reminiscent of the weak product of two edges: $| \times - = \times$. In this paper "graph product" will always mean the weak product.

In the first part of the paper we consider the interesting special case of the product of complete graphs on $r > 2$ vertices,

$$G = K_r^n = \times_{j=1}^n K_r.$$

We then discuss a more general setting, considering other r-regular graphs as well.

When $G = K_r^n$, we identify the vertices of G in the obvious way with the elements of $\mathbb{Z}_r^n$. Recalling the definition of the product, two vertices are adjacent in G iff the corresponding vectors differ in every coordinate. Clearly one can color G with r colors by choosing a coordinate i and coloring every vertex according to its ith coordinate. The following theorem asserts that if $r > 2$ then these are the only r-colorings. Here, and in what follows, we denote by $|H|$ the number of vertices of a graph H.

Theorem 1 *Let $G = K_r^n$, and assume $r \geq 3$. Let I be an independent set with $|I| = |G|/r$. Then there exists a coordinate $i \in \{1 \ldots n\}$ and $k \in \{0 \ldots r-1\}$ such that*

$$I = \{v : v_i = k\}.$$

Consequently, the only colorings of G by r colors are those induced by colorings of one of the factors K_r.

Greenwell and Lovász [2] proved the above theorem (and actually, a somewhat stronger statement) more than a quarter of a century ago. The novelty in this paper is the proof we supply that uses Fourier analysis on the group $\mathbb{Z}_r^n$. Our approach also allows us to deduce a stability version of the above theorem:

Theorem 2 *For every $r \geq 3$ there exists a constant $M = M(r)$ such that for any $\epsilon > 0$ the following is true. Let $G = K_r^n$. Let J be an independent set such that $\frac{|J|}{|G|} = \frac{1}{r} - \epsilon$. Then there exists an independent set I with $\frac{|I|}{|G|} = \frac{1}{r}$ such that $\frac{|J \triangle I|}{|G|} < M\epsilon$.*

Here "$\triangle$" denotes the symmetric difference. What the above theorem tells us is (in conjunction with Theorem 1) that any independent set that is *close* to being of maximum-size is close to being determined by one coordinate. We do not know of any purely combinatorial proof of this result.

The results in both theorems above can be extended to other base graphs. Let $\alpha(G)$ denote the maximum possible size of an independent set in a graph G. The following observation determines $\alpha(H^n)$ for any vertex transitive base graph H, in terms of $\alpha(H)$ and $|H|$.

Proposition 3 *For any vertex transitive graph H and for any integer $n \geq 1$, if $G = H^n$ then*

$$\frac{\alpha(G)}{|G|} = \frac{\alpha(H)}{|H|}.$$

After the simple proof of this proposition (some special cases of which are proved in [1]), we will provide some examples showing that the above equality does not necessarily hold without the transitivity assumption.

The relevance of graph eigenvalues to independent sets in graphs is well known and can be traced back to the old result that the independence number of any regular graph H on r vertices in which the eigenvalues of the adjacency matrix are $\mu_1 \geq \mu_2 \cdots \geq \mu_r$, is at most $-r\mu_r/(\mu_1 - \mu_r)$. A proof of this fact, as well as of the related results on the connection between the Shannon capacity of a graph and its eigenvalues, can be found in [3]. This bound is tight for many graphs H including, for example, complete graphs and the Petersen graph. It turns out that the results in Theorem 1 and in Theorem 2 can be extended to any connected non-bipartite regular base graph H for which the above bound is tight.

Theorem 4 *Let H be a connected d-regular graph on r vertices and let $d = \mu_1 \geq \mu_2 \geq \cdots \geq \mu_r$ be its eigenvalues. If*

$$\frac{\alpha(H)}{r} = \frac{-\mu_r}{d - \mu_r} \tag{1}$$

then for every integer $n \geq 1$,

$$\frac{\alpha(H^n)}{r^n} = \frac{-\mu_r}{d - \mu_r}.$$

Moreover, if H is also non-bipartite, and if I is an independent set of size $\frac{-\mu_r}{d-\mu_r} r^n$ in $G = H^n$, then there exists a coordinate $i \in \{1, 2, \ldots, n\}$ and a maximum independent set J in H, such that

$$I = \{v \in V(H)^n : \; v_i \in J\}.$$

Remark: Note that for any H and n, $\chi(H^n) = \chi(H)$. If H satisfies the conditions of the last Theorem and if, in addition, $\chi(H) = \frac{r}{\alpha(H)}$ then every optimal coloring of H^n is induced by a coloring of one of the multiplicands, since it is a partition of H^n into maximum-size independent sets. Such a partition can only be consistent if each color class is induced by the same coordinate. The assumption $\chi(H) = \frac{r}{\alpha(H)}$ holds for many of the interesting classes of graphs to which Theorem 4 applies.

Theorem 5 *Let H be a d-regular, connected, non-bipartite graph on r vertices, let $d = \mu_1 \geq \mu_2 \geq \cdots \geq \mu_r$ be its eigenvalues and suppose its independence number satisfies (1). Then, there exists a constant $M = M(H)$ such that for any $\epsilon > 0$ the following holds. Let $G = H^n$ and let I be an independent set such that $\frac{|I|}{|G|} = \frac{\alpha(H)}{|H|} - \epsilon$. Then there exists an independent set I' with $\frac{|I'|}{|G|} = \frac{\alpha(H)}{|H|}$ such that $\frac{|I' \triangle I|}{|G|} < M\epsilon$.*

References

[1] J. I. Brown, R. J. Nowakowski and D. Rall, The ultimate categorical independence ratio of a graph, SIAM J. Discrete Math. 9 (1996), 290-300.

[2] D. Greenwell, L. Lovász, Applications of Product Colorings, Acta Math. Acad. Sci. Hungar. 25 (3-4) (1974) pp. 335-340

[3] L. Lovász, *On the Shannon capacity of a graph*, IEEE Transactions on Information Theory IT-25, (1979), 1-7.

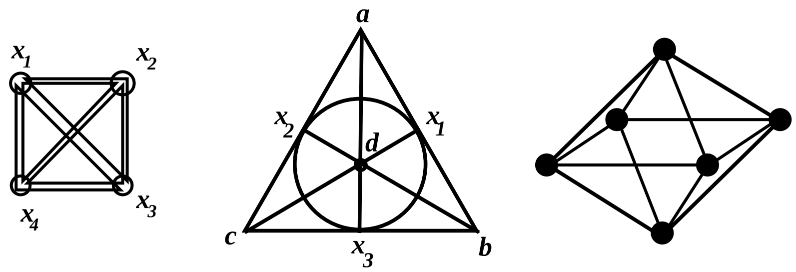
Triple Systems Not Containing a Fano Configuration and other Turán-type Problems

Zoltán Füredi

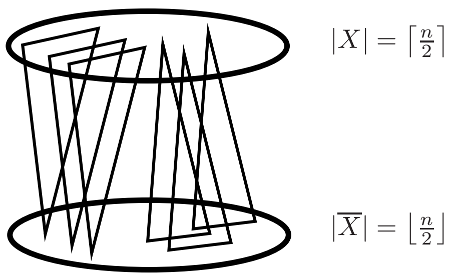
Given a 3-uniform hypergraph $\mathcal{F}$, let $\mathrm{ex}_3(n, \mathcal{F})$ denote the maximum possible size of a 3-uniform hypergraph of order n that does not contain any subhypergraph isomorphic to $\mathcal{F}$. The **Fano configuration** $\mathbb{F}$ (or Fano plane, or finite projective plane of order 2, or Steiner triple system, $STS(7)$, or blockdesign $S_2(7, 3, 2)$) is a hypergraph on 7 elements, say $\{x_1, x_2, x_3, a, b, c, d\}$, with 7 edges $\{x_1, x_2, x_3\}$, $\{x_1, a, b\}$, $\{x_1, c, d\}$, $\{x_2, a, c\}$, $\{x_2, b, d\}$, $\{x_3, a, d\}$, $\{x_3, b, c\}$. D. de Caen and Z. Füredi [2] proved a conjecture of Vera T. Sós [11] that

Theorem 1
$$\mathrm{ex}_3(n, \mathbb{F}) = \frac{3}{4}\binom{n}{3} + O(n^2).$$

The **tetrahedron**, $K_4^{(3)}$, i.e., a complete 3-uniform hypergraph on four vertices, has four triples $\{x_1, x_2, x_3\}$, $\{x_1, x_2, x_4\}$, $\{x_1, x_3, x_4\}$, $\{x_2, x_3, x_4\}$. An averaging argument shows [7] that the ratio $\mathrm{ex}_3(n, \mathcal{F})/\binom{n}{3}$ is a non-increasing sequence. Therefore
$$\pi(\mathcal{F}) := \lim_{n\to\infty} \mathrm{ex}_3(n, \mathcal{F})/\binom{n}{3}$$
exists. The determination of $\pi(K_4^{(3)})$ is one of the oldest problems of this field, due to Turán [12], who published a conjecture in 1961 that this limit value is 5/9, and Erdős [4] offered \$1000 for a proof. The best upper bound, .5935 . . . , is due to Fan Chung and Linyuan Lu [3]. The limit $\pi(\mathcal{H})$ is known only for very few cases when it is non-zero.



The Complete 4-graph, the Fano hypergraph, and the Octahedron



T. Sós also conjectured that the following hypergraph, $\mathcal{H}^n$, gives the exact value of $\mathrm{ex}_3(n, \mathbb{F})$. Let $\mathcal{H}(X, \overline{X})$ be the hypergraph obtained by taking the union of two disjoint sets X and $\overline{X}$ as the set of vertices and define the edge set as the set of all triples meeting both X and $\overline{X}$. For $\mathcal{H}^n$ we take $|X| = \lceil n/2 \rceil$ and $|\overline{X}| = \lfloor n/2 \rfloor$, (i.e., they have nearly equal sizes). Then

$$e(\mathcal{H}^n) = \binom{n}{3} - \binom{\lfloor n/2 \rfloor}{3} - \binom{\lceil n/2 \rceil}{3}.$$

The chromatic number of a hypergraph $\mathcal{H}$ is the minimum p such that its vertex set can be decomposed into p parts with no edge contained entirely in a single part. It is well known and easy to check that the Fano plane is not two-colorable, its chromatic number is 3. Therefore $\mathbb{F} \not\subseteq \mathcal{H}(X, \overline{X})$. Thus $\mathcal{H}^n$ supplies the lower bound for $\mathrm{ex}_3(n, \mathbb{F})$ in Theorem 1, implying that $\pi(\mathbb{F}) \geq \frac{3}{4}$.

Theorem 2 (Füredi and Simonovits [6]) *There exist a $\gamma_2 > 0$ and an n_2 such that the following holds. If $\mathcal{H}$ is a triple system on $n > n_2$ vertices not containing the Fano configuration $\mathbb{F}$ and*

$$\deg(x) > \left(\frac{3}{4} - \gamma_2\right)\binom{n}{2}$$

holds for every $x \in V(\mathcal{H})$, then $\mathcal{H}$ is bipartite, $\mathcal{H} \subseteq \mathcal{H}(X, \overline{X})$ for some $X \subseteq V(\mathcal{H})$.

This result is a distant relative of the following classical theorem of Andrásfai, Erdős and T. Sós [1]. Let G be a triangle-free graph on n vertices with minimum degree $\delta(G)$. If $\delta(G) > \frac{2}{5}n$, then G is bipartite. The blow up of a five-cycle C_5 shows that this bound is the best possible.

Using the method of [2] Mubayi and Rödl [9] determined the limit π for a few more 3-uniform hypergraphs, for all of them $\pi = 3/4$. It is very likely that the extremal hypergraphs are 2-colorable in those cases, too.

Turán [12] also conjectured that the 2-colorable triple system $\mathcal{H}^n$ is the largest $K_5^{(3)}$-free hypergraph. Sidorenko [10] disproved this conjecture, in this sharp form, for odd values $n \geq 9$. But it is still conjectured that it is true for all even values and it seems that $\pi(K_5^{(3)}) = 3/4$ holds as well. However this question seems to be extremely difficult.

De Caen and Füredi [2] applied some multigraph extremal results of Füredi and Kündgen [5]. To prove Theorem 2 we use colored multigraph extremal results.

A corollary of Theorem 2, namely that $\mathcal{H}(X, \overline{X})$ is extremal, was proved independently and in a fairly similar way by Keevash and Sudakov [8]. Our Theorem 2 is stronger.

References

[1] Andrásfai, B., Erdős, P. and T. Sós, V. (1974) On the connection between chromatic number, maximal clique and minimal degree of a graph. *Discrete Math.* **8** 205-218.

[2] de Caen, D. and Füredi, Z. (2000) The maximum size of 3-uniform hypergraphs not containing a Fano plane. *J. Combin. Theory Ser. B* **78** 274–276.

[3] Chung, F. R. K. and Lu, Linyuan (1999) An upper bound for the Turán number $t_3(n, 4)$. *J. Combin. Theory Ser. A* **87** 381–389.

[4] Erdős, P. (1981) On the combinatorial problems which I would most like to see solved. *Combinatorica* **1** 25–42.

[5] Füredi Z. and Kündgen, A. (2002) Turán problems for integer-weighted graphs. *J. Graph Theory* **40** 195–225.

[6] Füredi Z. and Simonovits, M. (2004) Triple systems not containing a Fano configuration. *Combin. Prob. and Comput.* to appear.

[7] Katona, G., Nemetz, T. and Simonovits, M. (1964) On a problem of Turán in the theory of graphs. *Mat. Lapok* **15** 228–238.

[8] Keevash, P. and Sudakov, B. The exact Turán number of the Fano plane. To appear in *Combinatorica*.

[9] Mubayi, D. and Rödl, V. (2002) On the Turán number of triple systems. *J. Combin. Th. Ser. A* **100** 136–152.

[10] Sidorenko, A. (1995) What we know and what we do not know about Turán numbers. *Graphs and Combin.* **11** 179–199.

[11] T. Sós, V. (1976) Some remarks on the connection of graph theory, finite geometry and block designs. *in:* Teorie Combinatorie, Tomo II., Accad. Naz. Linzei, Roma, pp. 223–233.

[12] Turán, P. (1961) Research problems. *MTA Mat. Kutató Int. Közl.* **6** 417-423.

Entropy and Graph Homomorphisms

David Galvin

(joint work with Prasad Tetali [3])

Let G be an n-regular, N-vertex bipartite graph on vertex set $V(G)$, and let H be a fixed graph on vertex set $V(H)$ (perhaps with loops). Set

$$Hom(G, H) = \{f : V(G) \to V(H) \;:\; u \sim v \Rightarrow f(u) \sim f(v)\}.$$

That is, $Hom(G, H)$ is the set of graph homomorphisms from G to H.

When $H = H_{ind}$ consists of one looped and one unlooped vertex connected by an edge, an element of $Hom(G, H_{ind})$ can be thought of as a specification of an independent set (a set of vertices spanning no edges) in G. Our point of departure is the following result of Kahn [4], bounding the size of $\mathcal{I}(G)$, the set of independent sets of G.

Theorem 1 *For any n-regular, N-vertex bipartite graph G,*

$$|\mathcal{I}(G)| \leq (2^{n+1} - 1)^{N/2n}.$$

Note that $|Hom(K_{n,n}, H_{ind})| = 2^{n+1} - 1$ (where $K_{n,n}$ is the complete bipartite graph with n vertices on each side), so we may paraphrase Theorem 1 by saying that $|Hom(G, H_{ind})|$ is maximum when G is a disjoint union of $K_{n,n}$'s. Our main result is a generalization of this statement (and our proof is a generalization of Kahn's).

Proposition 2 *For any n-regular, N-vertex bipartite G, and any H,*

$$|Hom(G, H)| \leq |Hom(K_{n,n}, H)|^{N/2n}.$$

We also consider a weighted version of Proposition 2. Following [1], we put a measure on $Hom(G,H)$ as follows. To each $i \in V(H)$ assign a positive "activity" λ_i, and write Λ for the set of activities. Give each $f \in Hom(G,H)$ weight $w^{\Lambda}(f) = \prod_{v \in V(G)} \lambda_{f(v)}$. The constant that turns this assignment of weights on $Hom(G,H)$ into a probability distribution is

$$Z^{\Lambda}(G,H) = \sum_{f \in Hom(G,H)} w^{\Lambda}(f).$$

When all activities are 1, we have $Z^{\Lambda}(G,H) = |Hom(G,H)|$, and so the following is a generalization of Proposition 2.

Proposition 3 *For any n-regular, N-vertex bipartite G, any H, and any system Λ of positive activities on $V(H)$,*

$$Z^{\Lambda}(G,H) \leq \left(Z^{\Lambda}(K_{n,n},H)\right)^{N/2n}.$$

We may put this result in the framework of a well-known mathematical model of physical systems with "hard constraints" (see [1]). We think of the vertices of G as particles and the edges as bonds between pairs of particles, and we think of the vertices of H as possible "spins" that particles may take. Pairs of bonded vertices of G may have spins i and j only when i and j are adjacent in H. Thus the legal spin configurations on the vertices of G are precisely the homomorphisms from G to H. We think of the activities on the vertices of H as a measure of the likelihood of seeing the different spins; the probability of a particular spin configuration is proportional to the product over the vertices of G of the activities of the spins. Proposition 3 concerns the "partition function" of this model — the normalizing constant that turns the above-described system of weights on the set of legal configurations into a probability measure.

Our proofs are based on entropy considerations, and in particular on a lemma of Shearer (see [2, p. 33]) bounding the entropy of a random vector.

References

[1] G. Brightwell and P. Winkler, Graph homomorphisms and phase transitions, *J. Combin. Theory Ser. B* **77** (1999), 221–262.

[2] F.R.K. Chung, P. Frankl, R. Graham and J.B. Shearer, Some intersection theorems for ordered sets and graphs, *J. Combin. Theory Ser. A.* **48** (1986), 23–37.

[3] D. Galvin and P. Tetali, On weighted graph homomorphisms, to appear in AMS volume on DIMACS/DIMATIA workshop *Graphs, Morphisms and Statistical Physics*, March 2001.

[4] J. Kahn, An entropy approach to the hard-core model on bipartite graphs, *Combin. Prob. Comp.* **10** (2001), 219–237.

Random Planar Graphs

Stefanie Gerke
(joint work with Colin McDiarmid [3])

Given $0 < p < 1$ and a positive integer n, let $G_{n,p}$ denote the random graph with nodes $v_1, \ldots, v_n$ in which the $\binom{n}{2}$ possible edges appear independently with probability p. We denote by $R_{n,p}$ the random graph $G_{n,p}$ conditioned on it being planar. (We may think of repeatedly sampling a graph $G_{n,p}$ until we find one that is planar.) Also, let us denote $R_{n,\frac{1}{2}}$ by R_n. Thus R_n is uniformly distributed over all labelled planar graphs on n nodes.

Rather little is known about random planar graphs, even about the number of edges in such graphs, which is our focus here. Let us denote the number of edges in a (simple) graph G by $m(G)$. Thus we are interested in the random variable $m(R_n)$ and more generally in $m(R_{n,p})$. Of course $m(G) \leq 3n - 6$ for any planar graph G on n nodes. The expected value $\mathbf{E}[m(R_n)]$ is at least $(3n-6)/2$ – see [2]. It is shown in [1] that $m(R_n) \leq 2.54n$ asymptotically almost surely (aas), that is with probability tending to 1 as $n \to \infty$. This result slightly improves the upper bound of 2.56 in [6]. We will show here in particular that $m(R_n) \geq \frac{13}{7}n + o(n)$ aas, thereby improving on the result from [2] mentioned above.

We now introduce two functions $f(\alpha)$ and $g(p)$ which are needed to state our two main results – see also Figure 1.

Given $1 < \alpha \leq 3$, let $k = k(\alpha) = \lfloor \frac{2\alpha}{\alpha - 1} \rfloor$, and let

$$f(\alpha) = \frac{1}{4}\left(k^2 + k + 6 - (k^2 - 3k + 6)\alpha\right).$$

It is not hard to verify that $f(\alpha)$ is continuous and decreasing on $1 < \alpha \leq 3$, and satisfies $f(\alpha) \to \infty$ as $\alpha \to 1$ and $f(3) = 0$, see also the end of Section 4. (The function f is also piecewise-linear and convex.) For $0 < p < 1$ we may define $g(p)$ to be the unique value $\rho \in (1,3)$ such that $f(\rho)/\rho = (1-p)/p$. The function g is continuous and increasing on $0 < p < 1$, and satisfies $g(p) \to 1$ as $p \to 0$, $g(\frac{1}{2}) = \frac{13}{7}$ and $g(p) \to 3$ as $p \to 1$. We are now able to state our theorem concerning the number of edges of random planar graphs.

Theorem 1 *Let $0 < p < 1$. Then as $n \to \infty$,*

$$\mathbf{E}[m(R_{n,p})] \geq g(p)n + o(n);$$

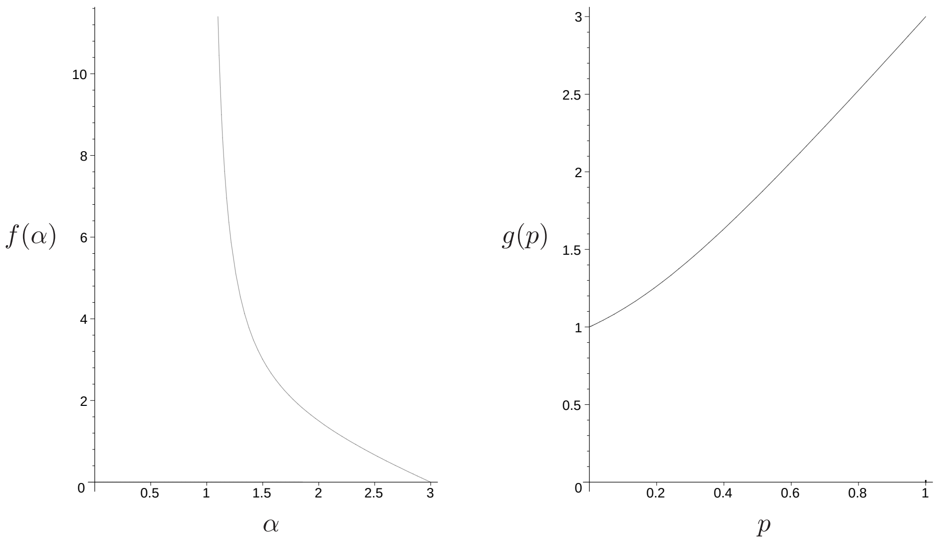


Figure 1: The functions f and g

and indeed for any $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that*

$$\mathbf{Pr}(m(R_{n,p}) < (g(p) - \varepsilon)n) = o(e^{-\delta n}).$$

In particular, since $g(\frac{1}{2}) = \frac{13}{7}$, this theorem shows that the expected number of edges in a planar graph sampled uniformly at random from all labelled planar graphs on n nodes is at least about $\frac{13}{7}n$.

To prove this result we will consider the number of edges that can be added to a planar graph of n nodes and m edges while keeping the graph planar. Given a planar graph G, we call a non-edge f *addable* in G if the graph $G + f$ obtained by adding f as an edge is still planar; and we let $\text{add}(G)$ denote the set of addable non-edges of G. Let $\mathcal{P}(n)$ denote the set of all (simple) planar graphs with n nodes $v_1, \ldots, v_n$; let $\mathcal{P}(n, m)$ denote the set of all graphs $G \in \mathcal{P}(n)$ with m edges; and let $\text{add}(n, m)$ denote the minimum value of $|\text{add}(G)|$ over all graphs $G \in \mathcal{P}(n, m)$. Observe that by Kuratowski's theorem, if $m \leq 7$ then $\text{add}(n, m) = \binom{n}{2} - m$, and if $n \geq 6$ and $m \geq 8$ then $\text{add}(n, m) < \binom{n}{2} - m$. Also, $\text{add}(n, m) > 0$ if $m < 3n - 6$ and $\text{add}(n, 3n - 6) = 0$.

Theorem 2 *Let* $1 < \alpha \leq 3$*, and suppose that* $m = m(n) = \alpha n + O(1)$ *as* $n \to \infty$*. Then* $\text{add}(n, m) = f(\alpha)n + O(1)$*.*

It was shown in [5] that a.a.s. the random planar graph contains any fixed connected planar graph. If one chooses a graph uniformly at random from $\mathcal{P}(n, m)$ with $m = \lfloor qn \rfloor$ $1 < q < 3$ then the same statements holds:

Theorem 3 (G., McDiarmid, Steger, Weißl [4]) *Let $1 < q < 3$. Then a.a.s. the random planar graph on n nodes and $\lfloor qn \rfloor$ edges contains any fixed connected planar graph.*

References

[1] N. Bonichon, C. Gavoille and Nicolas Hanusse, *An information-theoretic upper bound on planar graphs using triangulation,* in Proc. of the 20th Annual Symp. of Theoret. Comp. Science (2003) 499-510.

[2] A. Denise, M. Vasconcellos and D. Welsh, *The random planar graph,* Congr. Numer., **113** (1996) 61–79.

[3] S. Gerke and C. McDiarmid, On the number of edges in random planar graphs, Combin. Probab. Comput., to appear.

[4] S. Gerke, C. McDiarmid, A. Steger and A. Weissl, Random planar graphs with a fixed number of edges, manuscript.

[5] C. McDiarmid, A. Steger and D. Welsh, *Random planar graphs*, manuscript.

[6] D. Osthus, H.J. Prömel and A. Taraz, *On random planar graphs, the number of planar graphs and their triangulations,* J. Combin. Theory Ser. B, **88**(1) (2003) 119–134.

Low-dimensional Faces of Random 0/1-Polytopes

Volker Kaibel

Investigations of special classes of 0/1-polytopes (convex hulls of subsets of $\{0,1\}^d$) have not only lead to beautiful insights into combinatorial (optimization) problems during the last decades, but also powerful algorithms have emerged from them. Consequently, there has been some desire to learn more about the geometrical and combinatorial structure of 0/1-polytopes in general. Here, the study of random 0/1-polytopes has turned out to be particularly fruitful,

A quite fascinating result in this direction has been obtained by Dyer, Füredi, and McDiarmid in 1992, who proved in [2] that the expected volume $\mathbb{E}\,[\mathrm{Vol}\; P]$ of a d-dimensional random 0/1-polytope P with n vertices has a threshold at $2^{(1-(\log e)/2)d}$ (i.e., for each $\varepsilon > 0$, $\mathrm{Vol}\; P = \mathrm{o}(1)$ if $n \leq 2^{(1-(\log e)/2-\varepsilon)d}$ and $\mathrm{Vol}\; P =$

$1 - o(1)$ if $n \geq 2^{(1-(\log e)/2+\varepsilon)d}$). Building on the methods developed in Dyer, Füredi, and McDiarmid' work, Bárány and Pór proved in 2000 that a random 0/1-polytope (within a certain range of vertex numbers) has a super-exponential (in the dimension) number of facets [1].

While Bárány and Pór's work sheds some light on the highest dimensional faces of 0/1-polytopes, in my recent work (partly together with Anja Remshagen) I have investigated the lowest dimensional faces of random 0/1-polytopes. In [4] we proved that the expected graph density of a d-dimensional random 0/1-polytope P with n vertices has a threshold at $2^{(1/2)d}$. In [3] this result has been extended to the density of arbitrary (fixed) dimensional faces in the following way.

Denote by $\nu_r(P)$ the quotient of the number of faces of P with exactly r vertices and $\binom{n}{r}$ (the *r-density* of P). In [3], for each $r \geq 3$, we establish the existence of a sharp threshold for the r-density and determine the values of the threshold numbers τ_r such that, for all $\varepsilon > 0$,

$$\mathbb{E}\left[\nu_r(P)\right] = \begin{cases} 1 - o(1) & \text{if } n \leq 2^{(\tau_r - \varepsilon)d} \text{ for all } d \\ o(1) & \text{if } n \geq 2^{(\tau_r + \varepsilon)d} \text{ for all } d \end{cases}$$

holds for the expected value of $\nu_r(P)$.

In particular, these results indicate that the high densities often encountered in polyhedral combinatorics (e.g., the cut-polytope of the complete graph has both 2- and 3-density equal to one) are due to the geometry of 0/1-polytopes rather than to the special combinatorics of the underlying problems.

The threshold values τ_r (for $r \geq 3$) nicely extend the results for $r = 2$, while the proof becomes more involved and needs a heavier machinery (the one developed in the above mentioned paper by Dyer, Füredi, and McDiarmid). As a pay-back, however, it reveals several interesting insights into the geometry of (random) 0/1-polytopes.

References

[1] Imre Bárány and Attila Pór, *On* 0-1 *polytopes with many facets*, Adv. Math. **161** (2001), no. 2, 209–228.

[2] Martin E. Dyer, Zoltan Füredi, and Colin McDiarmid, *Volumes spanned by random points in the hypercube*, Random Structures Algorithms **3** (1992), no. 1, 91–106.

[3] Volker Kaibel, *Low-dimensional faces of random 0/1-polytopes*, Technical Report, TU Berlin, arXiv: math.CO/0311393 (extended abstract, 10 pages +3 pages appendix).

[4] Volker Kaibel and Anja Remshagen, *On the graph-density of random 0/1-polytopes*, Approximation, Randomization, and Combinatorial Optimization (Proc. RANDOM 2003) (S. Arora, K. Jansen, J.D.P. Rolim, and A. Sahai, eds.), Lecture Notes in Computer Science, vol. 2764, Springer, 2003, pp. 318–328.

Excluded Subposets in the Boolean Lattice

Gyula O.H. Katona

Introduction. Let $[n] = \{1, 2, \ldots, n\}$ be a finite set, families $\mathcal{F}, \mathcal{G}$, etc. of its subsets will be investigated. If $\mathcal{F}$ is a family let f_i denote the number of its i-element members. Let P be a poset. The goal of the present investigations is to determine the maximum size of a family $\mathcal{F}$ (in $[n]$) which does not contain P as a (non-necessarily induced) subposet. This maximum is denoted by $\mathrm{La}(n, P)$.

The easiest example is the case when P consist of two comparable elements (subsets of [1]). Then we are actually looking for the largest family without inclusion. The well-known Sperner theorem ([6]) gives the answer, the maximum is $\binom{n}{\lfloor \frac{n}{2} \rfloor}$.

The following sharpening, the so called YBLM inequality ([8], [1], [4], [5]) is also important.

Theorem 1 *If $\mathcal{F}$ is a family of subsets of $[n]$ without inclusion then*

$$\sum_{i=0}^{n} \frac{f_i}{\binom{n}{i}} \leq 1$$

holds

We say that the distinct sets $A, B_1 \ldots, B_r$ form an *r-fork* if they satisfy $A \subset B_1, \ldots, B_r$.

The first result in this direction of generalizing the Sperner theorem was the following one ([3]).

Theorem 2 *Suppose that $\mathcal{F}$ contains no 2-fork. Then*

$$\binom{n}{\lfloor \frac{n}{2} \rfloor}\left(1 + \frac{1}{n} + o\left(\frac{1}{n}\right)\right) \leq |\mathcal{F}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}\left(1 + \frac{2}{n} + o\left(\frac{1}{n}\right)\right)$$

holds.

The case of $r+1$-forks was considered in [7].

Theorem 3 *Suppose that $\mathcal{F}$ contains no $r+1$-fork. Then*

$$\binom{n}{\lfloor \frac{n}{2} \rfloor}\left(1+\frac{r}{n}+o\left(\frac{1}{n}\right)\right) \le |\mathcal{F}| \le \binom{n}{\lfloor \frac{n}{2} \rfloor}\left(1+2\frac{r^2}{n}+o\left(\frac{1}{n}\right)\right)$$

holds.

Let us remark that the lower estimates in the previous results and in the new results of the next section are all based on a code construction of [2] and its generalizations.

New results. The second term of the upper estimate is too weak in Theorem 1.3. We were recently able to improve this result.

Theorem 4 (A. de Bonis, G.O.H. Katona) *Suppose that $\mathcal{F}$ contains no $r+1$-fork. Then*

$$\binom{n}{\lfloor \frac{n}{2} \rfloor}\left(1+\frac{r}{n}+o\left(\frac{1}{n}\right)\right) \le |\mathcal{F}| \le \binom{n}{\lfloor \frac{n}{2} \rfloor}\left(1+\frac{2r}{n}+o\left(\frac{1}{n}\right)\right).$$

This is best possible in the sense that the coding problem what is used in the construction contains an undecided multiplicative factor 2.

The proof of the upper bound in the above theorem is based on the following YBLM-type inequality.

Theorem 5 (A. de Bonis, G.O.H. Katona) *Suppose that $\mathcal{F}$ contains no $r+1$-fork ($0 < r$) and all members $F \in \mathcal{F}$ satisfy $|F| \le m$. Then*

$$\sum_{i=0}^{n} \frac{f_i}{\binom{n}{i}} \le 1 + \frac{r}{n-m+1}.$$

Let us now try to maximize the size of a family $\mathcal{F}$ containing no $r+s+1$ distinct members satisfying $A_1, \ldots, A_s \subset B_1, \ldots, B_{r+1}$. Let $P_{r+1,s}$ denote the poset with two levels, s element on the lower, $r+1$ elements on the upper level, every lower one is in relation with every upper one. It is easy to see that our condition can be formulated in the way that we are looking for the maximum number of the elements in the Boolean lattice of subsets of $[n]$ (defined by inclusion) without containing $P_{r+1,s}$ as a subposet.

Theorem 6 (A. de Bonis, G.O.H. Katona) *Suppose that $2 \le s, 2 \le r$ and $s \le r+1$ hold. Then*

$$\begin{aligned}\binom{n}{\lfloor \frac{n}{2} \rfloor}\left(2+\frac{r}{n}+o\left(\frac{1}{n}\right)\right) &\le \mathrm{La}(n, P_{r+1,s}) \\ &\le \binom{n}{\lfloor \frac{n}{2} \rfloor}\left(2+2\frac{r+s-2}{n}+o\left(\frac{1}{n}\right)\right).\end{aligned}$$

Surprisingly, we have an exact result in the case $s = 2, r = 1$.

Theorem 7 (A. de Bonis, G.O.H. Katona, K. Swanepoel) *If* $5 \leq n$ *and* $\mathcal{F}$ *contains no four distinct members* A_1, A_2, B_1, B_2 *such that* $A_i \subset B_j, i, j = 1, 2$ *then the maximum of* $|\mathcal{F}|$ *is the sum of the two largest binomial coefficients of order* n.

References

[1] B. Bollobás, On generalized graphs, *Acta Math. Acad. Sci. Hungar* **16**(1965), 447–452.

[2] R.L. Graham and H.J.A. Sloane, Lower bounds for constant weight] codes, *IEEE IT* **26** 37-43.

[3] G.O.H. Katona and T. Tarján, Extremal problems with excluded subgraphs in the n-cube, *Lecture Notes in Math.* **1018**, 84-93.

[4] Lubell, A short proof of Sperner's lemma, *J. Combin. Theory* **1**(1966), 299.

[5] L.D. Meshalkin, A generalization of Sperner's theorem on the number of a finite set (in Russian), *Teor. Verojatnost. i Primen.* **8**(1963), 219–220.

[6] E. Sperner, Ein Satz über Untermegen einer endlichen Menge, *Math. Z.* **27**(1928) 544–548.

[7] Hai Tran Thanh, An extremal problem with excluded subposets in the Boolean lattice, *Order* **15**(1998) 51-57.

[8] K. Yamamoto, Logarithmic order of free distributive lattices, *J. Math. Soc. Japan* **6**(1954) 347–357.

Local Chromatic Number and Sperner Capacity

János Körner

(joint work with Concetta Pilotto and Gábor Simonyi)

Colouring the vertices of a graph so that no adjacent vertices receive identical colours gives rise to many interesting problems and invariants. The best known among all these invariants is the chromatic number, the minimum number of colours needed for such proper colourings. the following interesting variant was introduced by Erdős, Füredi, Hajnal, Komjáth, Rödl, and Seress [5] (cf. also [7]).

Definition 1 ([5]) *The local chromatic number $\psi(G)$ of a graph G is the maximum number of different colours appearing in the closed neighbourhood of any vertex, minimized over all proper colourings of G. Formally,*

$$\psi(G) := \min_{c:\ V(G)\to N} \max_{v\in V(G)} |\{c(u) : u \in \Gamma_G(v)\}|,$$

where N is the set of natural numbers, $\Gamma_G(v)$, the closed neighborhood of the vertex $v \in V(G)$, is the set of those vertices of G that are either adjacent or equal to v and $c : V(G) \to N$ runs over those functions that are proper colourings of G.

It was proved in [5] that there exist graphs with $\psi(G) = 3$ and $\chi(G)$ arbitrarily large.

Throughout our paper [12] the present extended abstract is referring to, we are interested in chromatic invariants as upper bounds for the Shannon capacity of undirected graphs and its natural generalization Sperner capacity for directed graphs. We treat Shannon capacity in terms that are complementary to Shannon's own, (cf. [15], [14] and [9], [11]). In this language Shannon capacity describes the asymptotic growth of the clique number in the co-normal powers of a graph. Shannon proved (although in different terms) that the Shannon capacity $c(G)$ of a graph is upper bounded by its fractional chromatic number.

We show that $\psi(G)$ is bounded from below by the fractional chromatic number of G. This proves, among other things, that $\psi(G)$ is always an upper bound for the Shannon capacity $c(G)$ of G, but it is not a very useful one since it is always weaker than the fractional chromatic number itself. Thus the situation is rather different in the case of directed graphs. We introduce an analog of the local chromatic number for directed graphs and show that it is always an upper bound for the Sperner capacity of the digraph at hand. To illustrate the usefulness of this bound we apply it to show, for example, that an oriented odd cycle with at least two vertices with outdegree and indegree 1 always has its Sperner capacity equal to that of the single-edge graph K_2. We introduce fractional versions that further strengthen our bounds.

The definition of the directed version of $\psi(G)$ is straightforward.

Definition 2 *The local chromatic number $\psi_d(G)$ of a digraph G is the maximum number of different colours appearing in the closed out-neighbourhood of any vertex, minimized over all proper colourings of G. Formally,*

$$\psi_d(G) := \min_{c:\ V(G)\to N} \max_{v\in V(G)} |\{c(w) : w \in \Gamma^+_G(v)\}|$$

where N is the set of natural numbers, $\Gamma^+_G(v)$, the closed out-neighbourhood of the vertex $v \in V(G)$, is the set of those vertices $w \in V(G)$ that are either equal to v or else are endpoints of directed edges $(v, w) \in E(G)$, originated in v, and $c : V(G) \to N$ runs over those functions that are proper colourings of G.

Our main goal is to prove that $\psi_d(G)$ is an upper bound for the Sperner capacity of digraph G.

Definition 3 *For directed graphs $G = (V, E)$ and $H = (W, L)$, the co-normal (or disjunctive or OR) product $G \cdot H$ is defined to be the following directed graph:*

$$V(G \cdot H) = V \times W$$

and

$$E(G \cdot H) = \{((v, w), (v', w')) : (v, v') \in E \quad \text{or} \quad (w, w') \in L\}.$$

The n^{th} co-normal (or disjunctive or OR) power G^n of digraph G is defined as the n-fold co-normal product of G with itself, i. e., the vertex set of G^n is $V^n = \{\mathbf{x} = (x_1 \dots x_n) : x_i \in V\}$, while its edge set is defined as

$$E(G^n) = \{(\mathbf{x}, \mathbf{y}) : \exists i \ (x_i, y_i) \in E(G)\}.$$

(A pair (a, b) always means an oriented edge in this paper as opposed to undirected edges denoted by $\{a, b\}$.)

Definition 4 ([9])
A subgraph of a digraph is called a symmetric clique if its edge set contains all ordered pairs of vertices belonging to the subgraph and we denote the size (number of vertices) of the largest symmetric clique by $\omega_s(G)$. The (non-logarithmic) Sperner capacity of a digraph G is defined as

$$\sigma(G) = \sup_n \sqrt[n]{\omega_s(G^n)}.$$

It is obvious that Sperner capacity is a generalization of Shannon capacity. It is a true generalization in the sense that there exist digraphs the Sperner capacity of which is different from the Shannon capacity ($c(G)$ value) of its underlying undirected graph. Denoting by G both an arbitrary digraph and its underlying undirected graph, it follows from the definitions that $\sigma(G) \leq c(G)$ always holds. The smallest example with strict inequality in the previous relation is a cyclically oriented triangle, cf. [4], [3].

Shannon capacity is is difficult to determine, and it is unknown for many relatively small and simple graphs, for example, for all odd cycles of length at least 7. This shows that Sperner capacity cannot be easy to determine either. There is an interesting and important connection between Sperner capacity and extremal set theory, introduced in [13] and fully explored in [10]. Several problems of this flavour are also discussed in [11].

Alon [1] proved that for any digraph G

$$\sigma(G) \leq \min\{\Delta_+(G), \Delta_-(G)\} + 1$$

where $\Delta_+(G)$ is the maximum out-degree of the graph G and similarly $\Delta_-(G)$ is the maximum in-degree. The proof relies on a linear algebraic method similar to the one already used in [3] for a special case (cf. also [6] for a strengthening and cf. [2] for a general setup for this method in case of undirected graphs). We also use this method for proving the following stronger result.

Theorem 5

$$\sigma(G) \leq \psi_d(G).$$

We call an oriented cycle alternating if it has at most one vertex of outdegree 1. (In stating the following results we follow the convention that an oriented graph is a graph without oppositely directed edges between the same two points, while a general directed graph may contain such pairs of edges.) Clearly, in any oriented cycle the number of vertices of outdegree 2 equals the number of vertices of outdegree 0. Thus, in particular, a $2k+1$ length oriented odd cycle is alternating if it has k points of outdegree zero, k points of outdegree 2 and only 1 point of outdegree 1. It takes an easy checking that up to isomorphism there is only one orientation of C_{2k+1} which is alternating.

Theorem 6 *Let G be an oriented odd cycle that is not alternating. Then*

$$\sigma(G) = 2.$$

The Sperner capacity of an alternating odd cycle can indeed be larger than 2. This is obvious for C_3, where the alternating orientation produces a transitive clique of size 3. A construction proving that the Sperner capacity of the alternating C_5 is at least $\sqrt{5}$ is given in [8]. The construction is clearly best possible by the celebrated result of Lovász [14] showing $c(C_5) = \sqrt{5}$.

References

[1] N. Alon, On the capacity of digraphs, *European J. Combin.*, **19** (1998), 1–5.

[2] N. Alon, The Shannon capacity of a union, *Combinatorica*, **18** (1998), 301–310.

[3] A. Blokhuis, On the Sperner capacity of the cyclic triangle, *J. Algebraic Combin.*, **2** (1993), 123–124.

[4] R. Calderbank, P. Frankl, R. L. Graham, W. Li, L. Shepp, The Sperner capacity of the cyclic triangle for linear and non-linear codes, *J. Algebraic Combin.*, **2** (1993), 31–48.

[5] P. Erdős, Z. Füredi, A. Hajnal, P. Komjáth, V. Rödl, Á. Seress, Coloring graphs with locally few colors, *Discrete Math.*, **59** (1986), 21–34.

[6] E. Fachini and J. Körner, Colour number, capacity and perfectness of directed graphs, *Graphs Combin.*, **16** (2000), 389–398.

[7] Z. Füredi, Local colourings of graphs (Gráfok lokális színezései), manuscript in Hungarian, September 2002.

[8] A. Galluccio, L. Gargano, J. Körner, G. Simonyi, Different capacities of a digraph, *Graphs Combin.*, **10** (1994), 105–121.

[9] L. Gargano, J. Körner, U. Vaccaro, Sperner theorems on directed graphs and qualitative independence, *J. Combin. Theory Ser. A*, **61** (1992), 173–192.

[10] L. Gargano, J. Körner, U. Vaccaro, Capacities: from information theory to extremal set theory, *J. Combin. Theory Ser. A*, **68** (1994), 296–316.

[11] J. Körner, A. Orlitsky, Zero-error information theory, *IEEE Trans. Inform. Theory*, **44**, No. 6 (October 1998, commemorative issue), 2207–2229.

[12] J. Körner, C. Pilotto, G. Simonyi, Local chromatic number and Sperner capacity, *J. Combin. Theory, Ser. B*

[13] J. Körner, G. Simonyi, A Sperner–type theorem and qualitative independence, *J. Combin. Theory Ser. A*, **59** (1992), pp. 90–103,

[14] L. Lovász, On the Shannon capacity of a graph, *IEEE Trans. Inform. Theory*, **25** (1979), 1–7.

[15] C. E. Shannon, The zero-capacity of a noisy channel, *IRE Trans. Inform. Theory*, **2** (1956), 8–19.

On H-linked Graphs

Alexandr Kostochka

(joint work with Gexin Yu)

In this talk, we introduce the notion of H-linked graphs and find suffiicient minimum degree conditions for a graph to be H-linked. This improves known conditions for a graph to be k-ordered.

Let H be a graph. An *H-subdivision* in a graph G is a pair of mappings $f : V(H) \to V(G)$ and g: $E(H)$ into the set of paths in G such that:

(a) $f(u) \neq f(v)$ for all distinct $u, v \in V(H)$;

(b) for every $uv \in E(H)$, $g(uv)$ is an $f(u)f(v)$-path in G, and distinct edges map into internally disjoint paths in G.

Say that a graph G is *H-linked* if every injective mapping $f : V(H) \to V(G)$ can be extended to an H-subdivision in G. This is a natural generalization of k-linkage.

Recall that a graph is *k-linked* if for every list of $2k$ vertices

$$\{s_1, \dots, s_k, t_1, \dots, t_k\} ,$$

there exist internally disjoint paths $P_1, \dots, P_k$ such that each P_i is an s_i, t_i-path. From the definitions of k-linked and H-linked graphs, we immediately see that *a graph G is k-linked if and only if G is H-linked for every graph H with $|E(H)| = k$.*

It is known that to check that a graph on at least $2k$ vertices is k-linked it is enough to check only the lists $\{s_1, \dots, s_k, t_1, \dots, t_k\}$, where all s_i and t_i are distinct. Thus, *a graph G on at least $2k$ vertices is k-linked if and only if G is M_k-linked, where M_k is the matching with k edges.*

Let B_k denote the (multi)graph with 2 vertices and k parallel edges. By Menger's Theorem, *a graph G on at least $k + 1$ vertices is k-connected if and only if G is B_k-linked.*

A graph is *k-ordered*, if for every ordered sequence of k vertices, there is a cycle that encounters the vertices of the sequence in the given order. Let C_k denote the cycle of length k. Clearly, *a graph G is k-ordered if and only if G is C_k-linked.*

After Chartrand introduced the notion of k-ordered graphs, several authors (see, e.g., [4, 8, 7, 5]) studied sufficient degree conditions for a graph to be k-ordered. Recall that Dirac [2] found sufficient conditions for a simple graph G to be Hamiltonian in terms of the minimum degree, $\delta(G)$, and Ore [9] found similar conditions in terms of $\sigma_2(G)$, the minimum value of the sum $\deg(u) + \deg(v)$ over all pairs $\{u, v\}$ of non-adjacent vertices in G. Let $D_0(n, k)$ denote the minimum positive integer d such that every n-vertex simple graph with minimum degree at least d is k-ordered. Similarly, let $R_0(n, k)$ denote the minimum positive integer r such that every n-vertex simple graph G with $\sigma_2(G) \geq r$ is k-ordered. Improving on results in [4, 8], it was shown in [5] that $R_0(n, k) = n + \lceil (3k - 9)/2 \rceil$ for every $3 \leq k \leq n/2$. This implies that $D_0(n, k) \leq \lceil (2n + 3k - 9)/4 \rceil$ for every $3 \leq k \leq n/2$. Moreover, Kierstead et al. [7] showed that $D_0(n, k) = \lceil \frac{n}{4} \rceil + \lfloor \frac{k}{2} \rfloor - 1$ for $3 \leq k \leq \frac{n+3}{11}$. Observe that these bounds demonstrate the interesting phenomenon: $R_0(n, k) > 2D_0(n, k)$ for k small with respect to n. It is also known that $D_0(n, k) > \lceil \frac{n}{4} \rceil + \lfloor \frac{k}{2} \rfloor - 1$ for $k > n/3$, but the value of $D_0(n, k)$ was not known for $\frac{n+3}{11} < k < \frac{2n}{5}$.

The main result of the talk gives the minimum degree conditions for a graph to be H-linked if $\delta(H) \geq 2$. This results extends the result of Kierstead et al. [7] in two directions: for a larger scope of k and for much more general H.

Theorem 1 *Let H be a simple graph with k edges and $\delta(H) \geq 2$. Every graph G of order $n \geq 5k$ with $\delta(G) \geq \lceil (n+k)/2 \rceil - 1$ is H-linked. If H is the cycle C_k with k edges, then every graph G of order $n \geq 5k$ with $\delta(G) \geq \lceil n/2 \rceil + \lfloor k/2 \rfloor - 1$ is H-linked. The minimum degree conditions are sharp.*

In particular, Theorem 1 yields $D_0(n,k) = \lceil \frac{n}{4} \rceil + \lfloor \frac{k}{2} \rfloor - 1$ for $k \leq n/5$.

Note that $\delta(G) \geq \lceil (n+k)/2 \rceil - 1$ is exactly the minimum degree condition that provides the k-connectivity of G. Thus, an evident degree condition for a graph to be k-connected, provides that a graph is H-linked for many H. If one drops the condition $\delta(H) \geq 2$, then this degree restriction is not sufficient in general. In a joint work with Kawarabayashi [6], we considered similar problem for k-linked graphs. Let $D(n,k)$ be the minimum positive integer d such that every n-vertex graph with minimum degree at least d is k-linked. Also, let $R(n,k)$ denote the minimum positive integer r such that every n-vertex graph G with $\sigma_2(G) \geq r$ is k-linked.

Theorem 2 [6] *If $k \geq 2$, then*

$$R(n,k) = \begin{cases} 2n-3, & n \leq 3k-1; \\ \lfloor \frac{2(n+5k)}{3} \rfloor - 3 & 3k \leq n \leq 4k-2; \\ n+2k-3, & n \geq 4k-1, \end{cases} \tag{1}$$

and

$$D(n,k) = \left\lceil \frac{R(n,k)}{2} \right\rceil = \begin{cases} n-1, & n \leq 3k-1; \\ \lfloor \frac{n+5k}{3} \rfloor - 1 & 3k \leq n \leq 4k-2; \\ \lceil \frac{n-3}{2} \rceil + k, & n \geq 4k-1. \end{cases} \tag{2}$$

Egawa et al. [3] considered a closely related problem, but the answers differ, especially for $\sigma_2(G)$. The bounds of Theorem 2 and of Egawa et al. [3] are helpful in estimating $f(k)$ — the minimum positive integer f such that every f-connected graph is k-linked. After a series of papers by Jung, Larman and Mani, Mader, and Robertson and Seymour, the first linear upper bound for f, namely, $f(k) \leq 22k$ was proved by Bollobás and Thomason [1]. Very recently, Thomas and Wollan [10] improved this bound to $f(k) \leq 16k$. In [6] we show how to apply Theorem 2 in the Thomas-Wollan proof to improve their bound to $f(k) \leq 12k$. Thomas and Wollan informed us that elaborating our idea they are able to improve the bound even further: to $f(k) \leq 10k$.

References

[1] B. Bollobás and A. Thomason, Highly linked graphs, *Combinatorica*, **16** (1996), 313–320.

[2] G. Dirac, Some theorems on abstract graphs, *Proc. London Math. Soc.*, **2** (1952), 69–81.

[3] Y. Egawa, R. J. Faudree, E. Györi, Y. Ishigami, R. H. Schelp, and H. Wang, Vertex-disjoint cycles containing specified edges, *Graphs and Combinatorics*, **16** (2000), 81–92.

[4] J. Faudree, R. Faudree, R. Gould, M. Jacobson, L. Lesniak, On k-ordered graphs, *J. Graph Theory*, **35** (2000), 69–82.

[5] R. J. Gould, A. V. Kostochka, L. Lesniak, I. Schiermeyer, and A. Saito, Degree conditions for k-ordered hamiltonian graphs, *J. Graph Theory*, **42** (2003), 199–210.

[6] K. Kawarabayashi, A. Kostochka, and G. Yu, On sufficient degree conditions for a graph to be k-linked, submitted.

[7] H. A. Kierstead, G. Sárközy, S. Selkow, On k-ordered hamiltonian graphs, *J. Graph Theory*, **32** (1999), 17–25.

[8] L. Ng and M. Schultz, k-Ordered hamiltonian graphs, *J. Graph Theory*, **2** (1997), 45–57.

[9] O. Ore, Note on Hamilton circuits, *Amer. Math. Monthly*, **67** (1960), 55.

[10] R. Thomas, P. Wollan, An Improved Linear Edge Bound for Graph Linkage, 2003, submitted.

Spanning Triangulations in Graphs with Large Minimum Degree

Daniela Kühn

(joint work with Deryk Osthus)

In [6] and [5] we investigated the following extremal problem: given a function $m = m(n)$, how large does the minimum degree of a graph G of order n have to be in order to guarantee a planar subgraph with at least $m(n)$ edges? The main result of [5] determines the minimum degree which is necessary to force a planar subgraph with the maximum possible number of edges, i.e. a planar triangulation.

Theorem 1 *There exists an integer n_0 such that every graph G of order $n \geq n_0$ and minimum degree at least $2n/3$ contains a triangulation as a spanning subgraph.*

Our proof of Theorem 1 can easily be extended to obtain a spanning triangulation of an arbitrary surface. Theorem 1 improves a result from [6] where the minimum degree was required to be at least $2n/3 + \gamma n$ (here $\gamma > 0$ can be chosen to be arbitrary small and $n_0 = n_0(\gamma)$).

The following example shows that Theorem 1 is best possible for all integers n which are divisible by 3. Consider the graph G^* which is obtained from two disjoint cliques A and B of order $n/3$ by adding an independent set C of $n/3$ new vertices and joining each of them to all the vertices in the two cliques. So G^* has minimum degree $2n/3 - 1$. Observe that any spanning triangulation in G^* would have two facial triangles T_1 and T_2 which share an edge and are such that T_1 contains a vertex of A and T_2 contains a vertex of B. But this is impossible since every triangle of G^* containing a vertex of A (respectively B) can have at most one vertex outside A (respectively B), namely in C. One can extend this example slightly to show that for all n a minimum degree of $\lceil 2n/3 \rceil - 1$ does not ensure a spanning triangulation (see [5]).

The spanning triangulation guaranteed by Theorem 1 can be found in polynomial time. In other words, the maximum planar subgraph problem (which in a given graph G asks for a planar subgraph with the maximum number of edges) can be solved in polynomial time for graphs G of minimum degree at least $2n/3$. In general this problem was shown to be Max SNP-hard by Călinescu et al. [1], i.e. there exists a positive ε for which there cannot be a polynomial time approximation algorithm with approximation ratio better than $1 - \varepsilon$, unless $P = NP$. The best known approximation algorithm has an approximation ratio of 4/9 [1].

Our proof of Theorem 1 relies on Szemerédi's Regularity lemma, the Blow-up lemma of Komlós, Sárközy and Szemerédi [4] and several ideas which were introduced in [3] by the same authors. (In [3] they proved the related result that every graph of sufficiently large order n and minimum degree at least $2n/3$ contains the square of a Hamilton cycle.)

In the remainder we discuss how Theorem 1 might perhaps be strengthened. Obviously a minimum degree of $2n/3$ will not force every given triangulation P of order n as a subgraph. For example, G might be 3-partite, which implies that we can only hope for triangulations P with chromatic number 3. Of course, we cannot guarantee all of these either, as there are triangulations whose chromatic number is 3 and whose maximum degree is $n - 2$. However, in view of our proof of Theorem 1, it might be helpful to restrict one's attention to triangulations P of bounded band-width, as this imposes a linear structure on P. (The *band-width* of a graph H is the smallest integer k for which there exists an enumeration $v_1, \ldots, v_{|H|}$ of the vertices of H such that every edge $v_i v_j \in H$ satisfies $|i - j| \leq k$.) Bollobás and Komlós [2] conjectured that for every $\gamma > 0$ and all $r, \Delta \in \mathbb{N}$ there are $\alpha > 0$

and $n_0 \in \mathbb{N}$ such that every graph G of order $n \geq n_0$ and minimum degree at least $(1 - \frac{1}{r} + \gamma)n$ contains a copy of every graph H of order n whose chromatic number is at most r, whose maximum degree is at most Δ and whose band-width is at most αn.

This would imply that every sufficiently large graph of minimum degree at least $(2/3 + \gamma)n$ contains every 3-chromatic triangulation of bounded band-width. Even in this special case the error term γn cannot be omitted completely: there are 3-chromatic triangulations whose colour classes have different sizes. These obviously do not embed into the complete 3-partite graph whose vertex classes have equal size. However, it might be true that for all integers b there exists a constant $C = C(b)$ such that every graph of order n and minimum degree at least $2n/3 + C$ contains every 3-chromatic triangulation of order n and band-width at most b as a subgraph.

Also, we do not know whether one can strengthen Theorem 1 in the following way. Given n, is there a triangulation P_n of order n which is contained in *every* graph G of order n and minimum degree at least $2n/3$? When n is divisible by 3, then the preceding arguments show that P_n would have to be 3-chromatic with equal size colour classes. Moreover, P_n would have to contain induced cycles of many different lengths. To see the latter, consider a graph G which is similar to the graph G^* defined earlier. This time the cliques have order $n/3 - 1$, the independent set C has order $n/3 + 2$ and we insert a 2-factor into C. One can show that every spanning triangulation of G must contain one of the cycles in $G[C]$.

References

[1] G. Călinescu, C.G. Fernandes, U. Finkler and H. Karloff, A better approximation algorithm for finding planar subgraphs, *J. Algorithms* **28** (1998), 105-124.

[2] J. Komlós, The Blow-up Lemma, *Comb. Probab. Comput.* **8** (1999), 161–176.

[3] J. Komlós, G. N. Sárközy and E. Szemerédi, On the square of a Hamilton cycle in dense graphs, *Random Struct. Alg.* **9** (1996), 193–211.

[4] J. Komlós, G. N. Sárközy and E. Szemerédi, Blow-up Lemma, *Combinatorica* **17** (1997), 109–123.

[5] D. Kühn and D. Osthus, Spanning triangulations in graphs, submitted.

[6] D. Kühn, D. Osthus and A. Taraz, Large planar subgraphs in dense graphs, submitted.

Revisiting Two Theorems of Curto and Fialkow on Moment Matrices

Monique Laurent

The moment problem

Given a probability measure μ on $\mathbb{R}^n$, the quantity $y_\alpha := \int x^\alpha \mu(dx)$ is called its *moment of order* α. The moment problem concerns the characterization of the sequences $y = (y_\alpha)_{\alpha\in\mathbb{Z}_+^n}$ that are the sequences of moments of some nonnegative measure μ; in that case one says that μ is a *representing measure* for y and μ is a probability measure if $y_0 = 1$. The results of Curto and Fialkow that we consider here deal with moment sequences of finite atomic measures, i.e., measures of the form $\mu = \sum_{i=1}^r \lambda_i \delta_{x_i}$ with $\lambda_1, \ldots, \lambda_r \neq 0$ and $x_1, \ldots, x_r \in \mathbb{R}^n$. Here, δ_x is the *Dirac measure* at $x \in \mathbb{R}^n$ (with mass 1 at x and 0 elsewhere), whose moment sequence is the *zeta vector* $\zeta_x := (x^\alpha)_{\alpha\in\mathbb{Z}^n} \in \mathbb{R}^{\mathbb{Z}_+^n}$.

Given $y \in \mathbb{R}^{\mathbb{Z}_+^n}$, its *moment matrix* is the symmetric matrix $M(y)$ indexed by $\mathbb{Z}_+^n$ whose (α,β)th entry is equal to $y_{\alpha+\beta}$, for $\alpha, \beta \in \mathbb{Z}_+^n$. A well known necessary condition for y to have a representing measure μ is the positive semidefiniteness of its moment matrix. Moreover, the support of μ is contained in the set of common zeros of the polynomials belonging to the kernel of $M(y)$ and the rank of $M(y)$ is at most the number of atoms in the support of μ.

The cone $\mathcal{M}$ consisting of the sequences y having a representing measure, and the cone $\mathcal{P}$ consisting of the polynomials nonnegative on $\mathbb{R}^n$, are dual of each other (Haviland [5]). Moreover, the cone $\mathcal{M}_+$ consisting of the sequences y whose moment matrix $M(y)$ is positive semidefinite, and the cone Σ^2 consisting of all sums of squares of polynomials, are dual of each other (Berg et al. [1]). Thus the moment problem can be cast - via duality - as the problem of characterizing nonnegative polynomials. The inclusion: $\Sigma^2 \subseteq \mathcal{P}$ is an equality for $n = 1$ and it is strict for $n \geq 2$, as already noticed by Hilbert in the 1890s. Equivalently, the inclusion: $\mathcal{M} \subseteq \mathcal{M}_+$ is an equality for $n = 1$ (this is Hamburger's theorem) and it is strict for $n \geq 2$.

There are, however, some cases when the implication: $y \in \mathcal{M}_+ \Longrightarrow y \in \mathcal{M}$ holds. Berg, Christensen and Ressel [1] show that this is true when y is bounded. Curto and Fialkow [2] show that this is true when $M(y)$ has finite rank.

Theorem 1 *[2] If $M(y) \succeq 0$ and $M(y)$ has finite rank r, then y has a unique representing measure, which is r-atomic.*

As a direct application of Theorem 1, the reverse implication also holds: If y has a r-atomic representing measure, then $M(y) \succeq 0$ and rank $M(y) = r$.

Curto and Fialkow's proof for Theorem 1 is along the following lines. (See chapter 4 in [2].) Assume $M(y) \succeq 0$ and rank $M(y) = r$. Then, the kernel $I := \{p \in \mathbb{R}[x_1, \ldots, x_n] \mid M(y)p = 0\}$ of $M(y)$ is an ideal in $\mathbb{R}[x_1, \ldots, x_n]$ and the quotient vector space $A := \mathbb{R}[x_1, \ldots, x_n]/I$ has finite dimension r. Define an inner product on A by setting $\langle p, q \rangle := p^T M(y) q$. In this way, A is a Hilbert space of dimension r. For $q \in \mathbb{R}[x_1, \ldots, x_n]$, consider the multiplication operator $\varphi_q : A \to A$ defined by $\varphi_q(p) = pq$. Obviously, the operators $\varphi_{x_1}, \ldots, \varphi_{x_n}$ pairwise commute. Curto and Fialkow use then the spectral theorem and the Riesz representation theorem for proving the existence of a representation measure for y. This type of proof based on functional analytic tools is often used for proving results about the moment problem. See, e.g., Fuglede [4], Schmüdgen [8].

The first main contribution of this paper is an alternative more elementary proof for Theorem 1. Our proof uses Hilbert's Nullstellensatz and, beside this algebraic result, it uses only elementary linear algebra. Our starting point is to observe that the kernel I of $M(y)$ is a *radical* ideal. Hence, the variety $V(I)$ (consisting of the common complex roots of all polynomials in I) has cardinality r. Say, $V(I) = \{v_1, \ldots, v_r\}$. Note that a complex point v belongs to $V(I)$ if and only if its conjugate $\overline{v}$ belongs to $V(I)$. Thus, one can write: $V(I) = S \cup T \cup \overline{T}$, where $S := V(I) \cap \mathbb{R}^n$ and $\overline{T} := \{\overline{v} \mid v \in T\}$.

Let $p_{v_1}, \ldots, p_{v_r} \in \mathbb{C}[x_1, \ldots, x_n]$ be interpolation polynomials at the points of $V(I)$; that is, $p_{v_i}(v_j) = 1$ if $i = j$ and $p_{v_i}(v_j) = 0$ if $i \neq j$, for $i, j = 1, \ldots, r$. One can assume that p_v is real valued for $v \in S$ and that $p_{\overline{v}} = \overline{p_v}$ for $v \in T$.

Let Z be the matrix whose columns are the zeta vectors $\zeta_{v_1}, \ldots, \zeta_{v_r}$, and let $\tilde{Z}$ be the matrix whose rows contain the coefficient vectors of the interpolation polynomials $p_{v_1}, \ldots, p_{v_r}$. Thus, $\tilde{Z} Z = I_r$. Theorem 1 now follows from the next three lemmas.

Lemma 2 $M(y) = Z \mathrm{diag}(\tilde{Z} y) Z^T$.

Lemma 3 $V(I) \subseteq \mathbb{R}^n$.

Lemma 4 $M(y) = \sum_{i=1}^r p_{v_i}^T M(y) p_{v_i} \zeta_{v_i} \zeta_{v_i}^T$ *and* $\mu := \sum_{i=1}^r p_{v_i}^T M(y) p_{v_i} \delta_{v_i}$ *is the unique measure representing* y.

The F-moment problem

Curto and Fialkow [3] study the F-moment problem for truncated sequences. That is, given a sequence $y \in \mathbb{R}^{S_{2t}}$, decide whether y has a representing measure supported by a given set $F \subseteq \mathbb{R}^n$. Here, for an integer $t \geq 1$, S_t denotes the set of $\alpha \in \mathbb{Z}_+^n$ with $\sum_i \alpha_i \leq t$. Consider the case when F is a basic closed semialgebraic set, of the form

$$F := \{x \in \mathbb{R}^n \mid h_1(x) \geq 0, \ldots, h_m(x) \geq 0\}, \tag{1}$$

where $h_1, \ldots, h_m \in \mathbb{R}[x_1, \ldots, x_n]$; set

$$d_j = \lceil \deg(h_j)/2 \rceil, \; d := \max_{j=1}^{m} d_j. \tag{2}$$

Necessary conditions can be formulated in terms of positive semidefiniteness of the localizing matrices of y. Given $h \in \mathbb{R}[x_1, \ldots, x_n]$, $h * y$ denotes the vector whose αth entry is $(h*y)_\alpha := \sum_\beta h_\beta y_{\alpha+\beta}$; its moment matrix is a *localizing matrix* of y. Moreover, $M_t(y)$ is the matrix indexed by S_t whose (α, β)th entry is $y_{\alpha+\beta}$. One can easily verify that, if $y \in \mathbb{R}^{S_{2t}}$ has a representing measure supported by the set F, then $M_t(y) \succeq 0$ and $M_{t-d_j}(h_j * y) \succeq 0$ for all $j = 1, \ldots, m$. Curto and Fialkow [3] show that, under certain rank assumptions, these necessary conditions are also sufficient for the existence of a representing measure supported by F. The following is the main result of [3] (Theorem 1.6 there).

Theorem 5 *[3] Let F be the set from (1) and let $d_1, \ldots, d_m, d$ be as in (2). Let $y \in \mathbb{R}^{S_{2t}}$ and $r := \text{rank } M_t(y)$. The following assertions are equivalent.*

(i) y has a r-atomic representing measure whose support is contained in F.

(ii) $M_t(y) \succeq 0$ and y can be extended to a vector $y \in \mathbb{R}^{S_{2(t+d)}}$ in such a way that $M_{t+d}(y)$ is a flat extension of $M_t(y)$ and $M_t(h_j * y) \succeq 0$ for $j = 1, \ldots, m$.

The second main contribution of our paper is a very short proof of this result. Assume (ii) holds. Then, by Theorem 6 below, y has a representing measure $\mu = \sum_{i=1}^r \lambda_i \delta_{v_i}$, where $r = \text{rank } M_t(y)$. Hence, it suffices to show that all v_i's belong to the set F. This follows from the assumption that $M_t(h_j * y) \succeq 0$, after observing that, as $r = \text{rank } M_t(y)$, one can find interpolation polynomials at $v_1, \ldots, v_r$ having degree at most t.

Theorem 6 *[2] Given $y \in \mathbb{R}^{S_{2t}}$, assume that $M_t(y) \succeq 0$ and that $\text{rank } M_t(y) = \text{rank } M_{t-1}(y)$. Then one can extend y to a vector in $\mathbb{R}^{\mathbb{Z}^n_+}$ having a representing measure which is $(\text{rank } M_t(y))$-atomic.*

Our study of the moment problem is partly motivated by its application to optimization. Indeed, Lasserre [7] shows how to construct asymptotic converging sequences of semidefinite relaxations using moment matrices, for the problem of minimizing a polynomial over a basic closed semi-algebraic set. Curto and Fialkow's results are used for proving, in some cases, the finite convergence. See also Henrion and Lasserre [6].

References

[1] C. Berg, J.P.R. Christensen, and C.U. Jensen. A remark on the multidimensional moment problem. *Math. Ann.* **243** (1979), 163–169.

[2] R.E. Curto and L.A. Fialkow. Solution of the truncated complex moment problem for flat data. *Mem. Amer. Math. Soc.* vol. 568, Amer. Math. Soc., Providence, RI, 1996.

[3] R.E. Curto and L.A. Fialkow. The truncated complex K-moment problem. *Trans. Amer. Math. Soc.* **352** (2000), 2825–2855.

[4] B. Fuglede. The multidimensional moment problem. *Expo. Math.* **1** (1983), 47–65.

[5] E.K. Haviland. On the momentum problem for distributions in more than one dimension. *Amer. J. Math.* **57** (1935), 562–568.

[6] D. Henrion and J.-B. Lasserre. Detecting global optimality and extracting solutions in GloptiPoly. Preprint, 2003.

[7] J.B. Lasserre. Global optimization with polynomials and the problem of moments. *SIAM J. Optim.* **11** (2001), 796–817.

[8] K. Schmüdgen. The K-moment problem for compact semi-algebraic sets. *Math. Ann.* **289** (1991), 203–206.

Partition Regular Equations

Imre Leader

(joint work with N. Hindman, P.A. Russell and D. Strauss)

An $n \times m$ matrix A, with rational entries, is called *partition regular* if whenever the natural numbers are finitely coloured there is a monochromatic vector x (meaning that all entries of x have the same colour) such that $Ax = 0$. We may also speak of the 'system of equations' $Ax = 0$ being partition regular.

The aim of this talk is to review some previous knowledge about the important notion of 'consistency', to be defined below, and then to go on to some more recent work. This recent work is joint with Hindman and Strauss [3],[4] and joint with Russell [5].

Many of the classical theorems of Ramsey Theory, such as Schur's theorem and van der Waerden's theorem, may naturally be interpreted at statements that certain matrices are partition regular. The partition regular matrices were characterised by Rado [7] in the 1930s. His characterisation had the following important consequence: if A and B are partition regular then so is their diagonal sum. In other words, if we can always solve $Ax = 0$ in one colour class, and we can always solve $By = 0$ in one colour class, then in fact we can solve $Ax = 0$ and $By = 0$ in the *same* colour class. We say that the matrices A and B are *consistent*.

This is important because it can be used to prove some 'universal' results. For example, whenever the natural numbers are finitely coloured, some class must contain solutions to *all* partition regular systems.

Let us now pass to the infinite case, where, in contrast to Rado's theorem, the whole picture is very much not yet understood. Which infinite systems of equations are partition regular? One very simple example, coming straight from Ramsey's theorem, is as follows: whenever the natural numbers are finitely coloured there exist $x_1, x_2, \ldots$ such that the set $\{x_i + x_j : i \neq j\}$ is monochromatic. (This is not quite given in the form of a solution to $Ax = 0$ for some suitable A, but it can easily be converted into that form if desired.) More generally, Ramsey's theorem implies that, for any fixed $a_1, \ldots, a_m$ positive integers, whenever the natural numbers are finitely coloured there exist $x_1, x_2, \ldots$ such that the set $\{a_1 x_{i_1} + \ldots + a_m x_{i_m} : i_1 < \ldots < i_m\}$ is monochromatic. We call this simple system a 'Ramsey' system. It is worth pointing out that one cannot relax the condition on $i_1, \ldots, i_m$ to the condition that they are merely distinct: for this system there are bad colourings.

The first non-trivial example of an infinite partition regular system was given by Hindman [2], who showed that whenever the natural numbers are finitely coloured there exist $x_1, x_2, \ldots$ such that the set

$$FS(x_1, x_2, \ldots) = \{\sum_{i \in I} x_i : 0 < |I| < \infty\}$$

is monochromatic. This was extended by Milliken [6]and Taylor [8], who showed that, for any fixed $a_1, \ldots, a_m$ positive integers, whenever the natural numbers are finitely coloured there exist $x_1, x_2, \ldots$ such that the set

$$FS_{a_1,\ldots,a_m}(x_1, x_2, \ldots) = \{a_1 \sum_{i \in I_1} x_i + \ldots + a_m \sum_{i \in I_m} x_i\}$$

is monochromatic, where we allow all finite nonempty $I_1, \ldots, I_m$ such that $\max I_r < \min I_{r+1}$ for all r. However, it is important to point out that not too many other examples of infinite partition regular systems are known.

It was proved by Deuber, Hindman, Leader and Lefmann [1] that unfortunately, in the infinite case, consistency does not always hold. Indeed, two different Milliken-Taylor systems are, except in trivial cases, always inconsistent. This left as a vexing open problem the question of whether or not the simple Ramsey systems were consistent. This was open for some time, being eventually solved by Hindman, Leader and Strauss [3]. The proof uses a large amount of machinery from the Stone-Cech compactification of the natural numbers (the space of ultrafilters), together with a new notion related to this space called 'central partition regularity'.

Recently, however, Leader and Russell [5] have found a very short proof of the consistency of Ramsey systems.

One other interesting development has concerned Ramsey systems with negative entries. Here we allow some of the a_i to be negative (although of course the final coefficient a_m must be positive, to have any hope of finding solutions in the natural numbers). One might imagine that this is just generalisation for its own sake, but curiously enough when one allows negative entries one suddenly obtains some much simpler proofs of inconsistency than were needed in [1]. This work is presented in [4].

References

[1] W. Deuber, N. Hindman, I. Leader and H. Lefmann, Infinite partition regular matrices, *Combinatorica* **15** (1995), 333-355.

[2] N. Hindman, Finite sums from sequences within cells of a partition of $\mathbb{N}$, *J. Combinatorial Theory (A)* **17** (1974), 1–11.

[3] N. Hindman, I. Leader and D. Strauss, Infinite partition regular matrices – solutions in central sets, *Trans. Amer. Math. Soc.* **355** (2003), 1213-1235.

[4] N. Hindman, I. Leader and D. Strauss, Separating Milliken-Taylor systems with negative entries, *Proc. Edinburgh Math. Soc.* **46** (2003), 45-61.

[5] I. Leader and P.A. Russell, Consistency for partition regular equations, submitted.

[6] K.R. Milliken, Ramsey's theorem with sums or unions, *J. Combinatorial Theory (A)* **18** (1975), 276-290.

[7] R. Rado, Studien zur Kombinatorik, *Math. Zeit.* **36** (1933), 242-280.

[8] A. Taylor, A canonical partition relation for finite subsets of ω, *J. Combinatorial Theory (A)* **21** (1976), 137-146.

Lifts, Discrepancy and Nearly Optimal Spectral Gaps

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(joint work with Yonatan Bilu)

Let G be a graph on n vertices. A 2-lift of G is a graph H on $2n$ vertices, with a covering map $\pi : H \to G$. It is not hard to see that all eigenvalues of G are also eigenvalues of H. In addition, H has n "new" eigenvalues. We conjecture that

every d-regular graph has a 2-lift such that all new eigenvalues are in the range $[-2\sqrt{d-1}, 2\sqrt{d-1}]$ (If true, this is tight , e.g. by the Alon-Boppana bound). Here we show that every graph of maximal degree d has a 2-lift such that all "new" eigenvalues are in the range $[-c\sqrt{d\log^3 d}, c\sqrt{d\log^3 d}]$ for some constant c. This leads to a polynomial time algorithm for constructing arbitrarily large d-regular graphs, with second eigenvalue $O(\sqrt{d\log^3 d})$.
The proof uses the following lemma (Lemma 5): Let A be a real symmetric matrix with zeros on the diagonal. Let d be such that the l_1 norm of each row in A is at most d. Let α be such that for every $x, y \in \{0,1\}^n$ with $< x, y >= 0$ it holds that $\frac{|xAy|}{||x||||y||} \leq \alpha$. Then the spectral radius of A is $O(\alpha(\log(d/\alpha)+1))$. An interesting consequence of this lemma is a converse to the Expander Mixing Lemma.

Definitions

Let $G = (V, E)$ be a graph on n vertices, and let A be its adjacency matrix. Let $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n$ be the eigenvalues of A. We say that G is an $(n, d, \mu)-$ *expander* if G is d-regular, and $\max_{i=2,\ldots,n} |\mu_i| \leq \mu$. When $\mu = 2\sqrt{d-1}$ we say that such a graph is *Ramanujan*. When $\mu = \tilde{O}(\sqrt{d})$ we say that such a graph is *Quasi-Ramanujan*.

A *signing* of the edges of G is a function $s : E(G) \to \{-1, 1\}$. The *signed adjacency matrix* of a graph G with a signing s has rows and columns indexed by the vertices of G. The (x, y) entry is $s(x, y)$ if $(x, y) \in E$ and 0 otherwise.
A 2-*lift* of G, associated with a signing s, is a graph $\hat{G}$ defined as follows. Associated with every vertex $x \in V$ are two vertices, x_0 and x_1, called the *fiber* of x. If $(x, y) \in E$, and $s(x, y) = 1$ then the corresponding edges in $\hat{G}$ are (x_0, y_0) and (x_1, y_1). If $s(x, y) = -1$, then the corresponding edges in $\hat{G}$ are (x_0, y_1) and (x_1, y_0). The graph G is called the *base graph*, and $\hat{G}$ a 2-lift of G. By the *spectral radius of a signing* we refer to the spectral radius of the corresponding signed adjacency matrix. When the spectral radius of a signing of a d-regular graph is $\tilde{O}(\sqrt{d})$ we say that the signing (or the lift) is *Quasi-Ramanujan*.

Quasi-Ramanujan 2-Lifts and Quasi-Ramanujan Graphs

Preliminaries. The eigenvalues of a 2-lift of G can be easily characterized in terms of the adjacency matrix and the signed adjacency matrix:

Lemma 1 *Let A be the adjacency matrix of a graph G, and A_s the signed adjacency matrix associated with a 2-lift $\hat{G}$. Then every eigenvalue of A and every eigenvalue of A_s are eigenvalues of $\hat{G}$. Furthermore, the multiplicity of each eigenvalue of $\hat{G}$ is the sum of its multiplicities in A and A_s.*

Consider the following scheme for constructing (n, d, λ)-expanders. Start with $G_0 = k_{d+1}$, the complete graph on $d + 1$ vertices * . Its eigenvalues are d, with multiplicity 1, and -1, with multiplicity d. We want to define G_i as a 2-lift of G_{i-1}, such that all new eigenvalues are in the range $[-\lambda, \lambda]$. Assuming such a 2-lifts always exist, the G_i constitute an infinite family of (n, d, λ)-expanders.
It is therefore natural to look for the smallest $\lambda = \lambda(d)$ such that every graph of degree at most d has a 2-lift, with new eigenvalues in the range $[-\lambda, \lambda]$. In other words, a signing with spectral radius $\leq \lambda$. We note that $\lambda(d) \geq 2\sqrt{d-1}$ follows from the Alon-Boppana bound.
Quasi-Ramanujan 2-lifts for every graph. Based on extensive computer simulations we conjecture that every graph has a signing with small spectral radius:

Conjecture 2 *Every d-regular graph has a signing with spectral radius at most $2\sqrt{d-1}$.*

In this section we show a close upper bound:

Theorem 3 *Every graph of maximal degree d has a signing with spectral radius $O(\sqrt{d \cdot \log^3 d})$.*

The theorem follows immediately from the following two lemmata (along with Lemma 1). The first shows that with positive probability the Rayliegh quotient is small for vectors in $v, u \in \{-1, 0, 1\}^n$. The second shows that this is essentially a sufficient condition for all eigenvalues being small.

Lemma 4 *For every graph of maximal degree d, there exists a signing s such that for all $v, u \in \{-1, 0, 1\}^n$ the following holds:*

$$\frac{|v^t A_s u|}{||v||||u||} \leq 10\sqrt{d \log d}, \tag{1}$$

where A_s is the signed adjacency matrix.

Lemma 5 *Let A be an $n \times n$ real symmetric matrix such that the l_1 norm of each row in A is at most d. Assume that for any two vectors, $u, v \in \{0, 1\}^n$, with $supp(u) \cap supp(v) = \emptyset$:*

$$\frac{|uAv|}{||u||||v||} \leq \alpha,$$

and that all diagonal entries of A are, in absolute value, $O(\alpha(log(d/\alpha)+1))$. Then the spectral radius of A is $O(\alpha(log(d/\alpha) + 1))$.

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*We could start with any small d-regular graph with a large spectral gap. Such graphs are easy to find.

Note 6 *Lemma 5 is tight up to constant factors. To see this, consider the n-dimensional vector x whose i'th entry is $1/\sqrt{i}$. Let A be the outer product of x with itself, that is, the matrix whose (i,j)'th entry is $1/\sqrt{i \cdot j}$. Clearly x is an eigenvector of A corresponding to the eigenvalue $||x||^2 = \Theta(\log(n))$. Also, the sum of each row in A is $O(\sqrt{n})$. To prove that the lemma is essentially tight, we need to show that $max_{u,v\in\{0,1\}^n} \frac{uAv}{||u||||v||}$ is constant. Indeed, fix $k,l \in [n]$. Let $u,v \in \{0,1\}^n$ be such that $||u|| = k$ and $||v|| = l$. As the entries of A are decreasing along the rows and the columns, uAv is maximized for such vectors when their support is the first k and l coordinates. For these optimal vectors, $uAv = \Theta(\sqrt{k \cdot l})$. Thus,*

$$max_{u,v\in\{0,1\}^n} \frac{uAv}{||u||||v||} = \Theta(1).$$

An explicit construction of quasi-Ramanujan graphs. For the purpose of building expanders, it is enough to prove a weaker version of Theorem 3. Roughly, that every *expander* graph has a 2-lift with small spectral radius. In this subsection we show that when the base graph is a good expander (in the sense of the definition below), then w.h.p. a random 2-lift has a small spectral radius. We then derandomize the construction to get a deterministic polynomial time algorithm for constructing arbitrarily large expander graphs.

Definition 7 *We say that a graph G on n vertices is (β, t)-sparse if for every $u, v \in \{0,1\}^n$, with $|S(u,v)| \leq t$,*

$$uAv \leq \beta ||u||||v||.$$

Lemma 8 *Let A be the adjacency matrix of a d-regular $(\gamma(d), \log n)$-sparse G graph on n vertices, where $\gamma(d) = 10\sqrt{d \log d}$. Then for a random signing of G (where the sign of each edge is chosen uniformly at random) the following hold w.h.p.:*

1. $\forall u, v \in \{-1,0,1\}^n : |uA_s v| \leq \gamma(d)||u||||v||$.
2. $\hat{G}$ *is* $(\gamma(d), 1 + \log n)$*-sparse*

where A_s is the random signed adjacency matrix, and $\hat{G}$ is the corresponding 2-lift.

Corollary 9 *Let A be the adjacency matrix of a d-regular $(\gamma(d), \log n)$-sparse G graph on n vertices, where $\gamma(d) = 10\sqrt{d \log d}$. Then there is a deterministic polynomial time algorithm for finding a signing s of G such that the following hold:*

1. *The spectral radius of A_s is $O(\sqrt{d \log^3 d})$.*
2. $\hat{G}$ *is* $(\gamma(d), 1 + \log n)$*-sparse,*

where A_s is the signed adjacency matrix, and $\hat{G}$ is the corresponding 2-lift.

A converse to the Expander Mixing Lemma

There are several approaches to expansion in graphs. A combinatorial definition says that a d-regular graph on n vertices is an (n, d, c)-vertex expander if every set of vertices, W, of size at most $n/2$, has at least $c|W|$ neighbors outside itself. An algebraic definition says that such a graph is an (n, d, λ)-expander if all eigenvalues but the largest are, in absolute value, at most λ.

The two notions are closely related. For example, it is known (cf. [2]) that an (n, d, λ)-expander is also an $(n, d, \frac{d-\lambda}{2d})$-vertex expander. Conversely, Alon shows in [1] that an (n, d, c)-vertex expander is also an $(n, d, d - \frac{c^2}{4+2c^2})$-expander. Roughly, these results show that one type of expansion implies the other. However, in all such results one implication (from combinatorial to algebraic expansion) is much weaker than the other.

For two subsets of vertices, S and T, let $e(S, T)$ denote the number of edges between them. A very useful property of (n, d, λ)-expanders is known as the Expander Mixing Lemma (cf. [2]): For every two subsets of vertices, A and B, of an (n, d, λ)-expander:

$$|e(A, B) - d|A||B|/n| \le \lambda\sqrt{|A||B|}.$$

Lemma 5 also implies a converse to this well known fact:

Corollary 10 *Let G be a d-regular graph on n vertices. Suppose that for any $S, T \subset V(G)$, with $S \cap T = \emptyset$*

$$|e(S, T) - \frac{|S||T|d}{n}| \le \alpha\sqrt{|S||T|}$$

Then all but the largest eigenvalue of G are bounded, in absolute value, by $O(\alpha(1 + \log(d/\alpha)))$.

It is known that for a random d-regular graph, w.h.p., the condition in Corollary 10 holds with $\alpha = O(\sqrt{d})$ (cf. [3]). Hence, it follows from the corollary that w.h.p., such a graph is an $(n, d, O(\sqrt{d}\log d))$-expander. While this result is weaker than previous ones ([6, 5, 4]), the proof here is somewhat shorter and simpler.

Acknowledgments. We thank László Lovász for insightful discussions, and Efrat Daom for help with computer simulations. We thank Eran Ofek for suggesting that Corollary 10 might be used to bound the second eigenvalue of random d-regular graphs.

We appreciate the helpful comments we got from Alex Samorodnitsky, Eyal Rozenman and Shlomo Hoory.

References

[1] N. Alon. Eigenvalues and expanders. *Combinatorica*, 6(2):83–96, 1986. Theory of computing (Singer Island, Fla., 1984).

[2] N. Alon and J. H. Spencer. *The probabilistic method*. Wiley-Interscience Series in Discrete Mathematics and Optimization. Wiley-Interscience [John Wiley & Sons], New York, second edition, 2000. With an appendix on the life and work of Paul Erdős.

[3] Y. Bilu and S. Hoory. On codes and hypergraphs. To appear in European Journal of Combinatorics.

[4] J. Friedman. A proof of alon's second eigenvalue conjecture. manuscript.

[5] J. Friedman. On the second eigenvalue and random walks in random d-regular graphs. *Combinatorica*, 11(4):331–362, 1991.

[6] J. Friedman, J. Kahn, and E. Szemeredi. On the second eigenvalue of random regular graphs. In *Proceedings of the twenty-first annual ACM symposium on Theory of computing*, pages 587–598. ACM Press, 1989.

Expected Length of the Longest Common Subsequence for Large Alphabets

Jiří Matoušek
(joint work with Marcos Kiwi and Martin Loebl)

We investigate the distribution of the length L of the longest common subsequence of two randomly uniformly and independently chosen n character words $u = u_1u_2 \dots u_n$ and $v = v_1v_2 \dots v_n$ over a k-ary alphabet. That is, L is the maximum integer such that there exist indices $i_1 < i_2 < \cdots < i_L$ and $j_1 < j_2 < \cdots < j_L$ with $u_{i_q} = v_{j_q}$, $q = 1, 2, \dots, L$. This problem has emerged more or less independently in several remarkably disparate areas, including the comparison of versions of computer programs, cryptographic snooping, and molecular biology. An extended abstract of this work appears in *Proc. 6th Latin American Theoretical Informatics Symposium* (LATIN 2004), LNCS series, Springer, Berlin. A full version is available at the web page of the author.

By a well-known subadditivity argument, $\mathbf{E}[L]/n$ converges to a constant γ_k. The value of γ_k is not known for any particular value of k, although much effort has been spent in finding good upper an lower bounds for it (see, for example, [2] and references therein).

We analyze the behavior of γ_k for $k \to \infty$, and more generally, we consider the expectation of L when k is an (arbitrarily slowly growing) function of n and

$n \to \infty$. In particular, we prove a conjecture of Sankoff and Mainville from the early 80's [7] stating that

$$\lim_{k\to\infty} \gamma_k \sqrt{k} = 2. \tag{1}$$

(See [6, § 6.8] for a discussion of work on lower and upper bounds on γ_k as well as a stronger version, due to Arratia and Steele, of the above stated conjecture.)

The constant 2 in (1) arises from a connection with another celebrated problem, the distribution of LIS_N, the length of the longest increasing subsequence in a (uniform) random permutation of $\{1, 2, .., N\}$. Hammersley [4] proved the existence of $\beta = \lim_{N\to\infty}(\mathbf{E}[\mathrm{LIS}_N]/\sqrt{N})$ and conjectured that $\beta = 2$. Later, Logan and Shepp [5], based on a result by Schensted [8], proved $\beta \leq 2$, and finally, Vershik and Kerov [10] showed $\beta = 2$. In a major recent breakthrough Baik, Deift, Johansson [3] described explicitly the asymptotic distribution of LIS_N (for $N \to \infty$). For a detailed account of these results, history, and related work we can recommend the surveys of Aldous and Diaconis [1] and Stanley [9]; the methods used in attacking this problem are of remarkable beauty and diversity.

Our main result about the longest common subsequence can be stated as follows.

Theorem 1 *For every $\varepsilon > 0$ there exist k_0 and C such that for all $k > k_0$ and all n with $n/\sqrt{k} > C$ we have*

$$(1-\varepsilon)\cdot\frac{2n}{\sqrt{k}} \;\leq\; \mathbf{E}[L] \;\leq\; (1+\varepsilon)\cdot\frac{2n}{\sqrt{k}}$$

where, as above, L is the length of the longest common subsequence of two random words of length n over an alphabet of size k. Moreover, there is an exponentially small tail bound; namely, for every $\varepsilon > 0$ there exists $c > 0$ such that for k and n as above,

$$\mathbf{P}\left[\left|L - \frac{2n}{\sqrt{k}}\right| \geq \varepsilon\frac{2n}{\sqrt{k}}\right] \leq e^{-cn/\sqrt{k}}.$$

In the rest of this extended abstract, we outline the main tools and ideas of the proof, referring to the full version for precise formulations and further details.

First we reformulate the problem a little. Given the two random words $u = u_1u_2\ldots u_n$ and $v = v_1v_2\ldots v_n$, let us draw two horizontal lines in the plane and place n points $a_1, a_2, \ldots, a_n$ in this order on the top line and n points $b_1, b_2, \ldots, b_n$ in this order on the bottom line. Then we connect a_i to b_j by an edge (straight segment) iff $u_i = v_j$, obtaining a drawing of a bipartite graph G (which is a disjoint union of complete bipartite graphs). A common subsequence of the words u and v corresponds to a *planar matching* in G (a matching in which no two edges cross).

Although we want to deal mainly with the case of n arbitrarily large compared to k, which is the situation in the Sankoff–Mainville conjecture, we first consider a seemingly opposite setting: when k is large and n is also large but considerably smaller than n. For definiteness, we set $n = k^{0.7}$. Then we expect G to have about $n^2/k = k^{0.4}$ edges, and most of these edges connect vertices of degree 1. If we let G' be the subgraph of G obtained by deleting all edges incident to vertices of degree greater than 1, then G' is a matching (plus some isolated vertices). The number N of edges of G' is typically very close to $k^{0.4}$. The matching determines a permutation of $\{1, 2, \ldots, N\}$, and by a symmetry argument, it can be seen that, for a given N, all permutations of $\{1, 2, \ldots, N\}$ have the same probability of being obtained in this way. Moreover, the longest increasing subsequence of the permutation corresponds exactly to the largest planar matching in G'. Therefore, up to a small error, the longest common subsequence of u and v is distributed as LIS_N. Then one can derive from the known results about LIS_N that $\mathbf{E}[L] = (2 + o(1))n/\sqrt{k}$ holds in this situation. For the rest of the proof, we also need tail estimates for large deviations of L, and these are conveniently obtained from Talagrand's inequality applied to L (we cannot directly use known tail estimates for LIS_N, for example because of the vertices of degree larger than 1 in G).

Now we consider n very large compared to k (and k still large). A lower bound for $\mathbf{E}[L]$ is straightforward: We partition the words u and v into segments of length $k^{0.7}$ each, and we use the previously derived result separately for each block (the ith block consists of the ith segment of the word u plus the ith segment of the word v). Thus, the lower bound is provided by a common subsequence, or planar matching in the graph language, that never crosses a block boundary.

An upper bound for $\mathbf{E}[L]$ is more demanding, since the largest planar matching need not respect any partition into blocks fixed in advance; there could be "very skew" edges. Our strategy is to simultaneously consider many different partitions into blocks. The blocks have upper and lower segments of size about $k^{0.7}$, but they can be very skew; the segment of u starting at a position i can form a block with a segment of v starting at position j, with i and j differing by a large amount. Supposing that there is a planar matching with at least $m = (1+\varepsilon)2n/\sqrt{k}$ edges, it "fits" at least one of the block partitions, meaning that it respects its block boundaries. For each fixed block partition and each fixed distribution of the numbers of edges of the planar matching among the blocks, we bound above the probability that there is a planar matching with m edges that fits that block partition; this relies on independence among the blocks. Then we sum up over all possible block partitions and show that with high probability, there is no planar matching with m edges at all.

References

[1] D. Aldous and P. Diaconis. Longest increasing subsequences: From patience sorting to the Baik–Deift–Johansson theorem. *Bulletin of the AMS*, 36(4):413–432, 1999.

[2] R. Baeza-Yates, G. Navarro, R. Gavaldá, and R. Schehing. Bounding the expected length of the longest common subsequences and forests. *Theory of Computing Systems*, 32(4):435–452, 1999.

[3] J. Baik, P. A. Deift, and K. Johansson. On the distribution of the length of the longest increasing subsequence of random permutaions. *J. Amer. Math. Soc.*, 12:1119–1178, 1999.

[4] J. M. Hammersley. A few seedlings of research. In *Proc. Sixth Berkeley Sympos. Math. Stat. Prob.*, pages 345–394, Berkeley, Calif., 1972. Univ. of California Press.

[5] B. F. Logan and L. A. Shepp. A variational problem or random Young tableaux. *Adv. in Math.*, 26:206–222, 1977.

[6] P.A. Pevzner. *Computational Molecular Biology: An Algorithmic Approach*. MIT Press, 2000.

[7] D. Sankoff and J. Kruskal, editors. *Common subsequences and monotone subsequences*, chapter 17, pages 363–365. Addison–Wesley, Reading, Mass., 1983.

[8] C. Schensted. Longest increasing and decreasing subsequences. *Canad. J. Math.*, 13:179–191, 1961.

[9] R. P. Stanley. Recent progress in algebraic combinatorics. *Bulletin of the AMS*, 40(1):55–68, 2002.

[10] A. M. Vershik and S. V. Kerov. Asymptotics of the Plancherel measure of the symmetric group and the limiting form of Young tableaux. *Dokl. Akad. Nauk SSSR*, 233:1024–1028, 1977.

On the Power of Two Choices in Continuous Time

Colin McDiarmid
(joint work with Malwina Luczak)

Balls-and-bins processes have been useful for analysing a wide range of problems, in discrete mathematics and computer science, and in particular for problems which involve load sharing and resource balancing, see [8]. Here is one central result, from Azar, Broder, Karlin and Upfal (1994 [1],1999 [2]), concerning the 'power of two choices'. Let d be a fixed positive integer. Suppose that there are n bins, and n balls arrive one after another: each ball picks d bins uniformly at random and is placed in a least loaded of these bins. Then with probability tending to 1 as $n \to \infty$ (*aas*), the maximum load of a bin is $\ln n/\ln\ln n + O(1)$ if $d = 1$, and is $\ln\ln n/\ln d + O(1)$ if $d \geq 2$. Thus there is a dramatic drop when we move from 1 to 2 choices.

In some recent work, balls have been allowed to 'die' – see [2, 3, 9] – which is of course desirable when modelling telephone calls. For example, suppose that we start with n balls in n bins: at each time step, one ball is deleted uniformly at random, and one new ball appears and is placed in one of d bins as before. It is shown in [2] that as $n \to \infty$, at any given time $t \geq cn^2 \ln\ln n$, aas the maximum load of a bin is at most $\ln\ln n/\ln d + O(1)$.

Let us consider here a simple and natural 'immigration-death' balls-and-bins model in continuous time. Indeed let us consider two such models, one involving bins and one involving queues, first the bins.

Let d be a fixed positive integer. Suppose that there are n bins. Balls arrive in a Poisson process at rate λn, where $\lambda > 0$ is a constant. Upon arrival each ball chooses d random bins (with replacement), and is placed into a least-loaded bin among those chosen. (If there is more than one chosen bin with least load, the ball is placed in the first such bin chosen.) Balls have independent exponential lifetimes with unit mean.

Probabilists have proved various detailed weak-convergence results for such models, see for example [4, 9, 10], but these results seem not to say anything about quantities like the equilibrium maximum load. Using mainly combinatorial methods, we can establish concentration results, which apply to the fraction of bins with load at least k at time t; these concentration results may then be used to analyse a balance equation involving these quantities. We are thus able to handle random variables like the maximum load, over long periods of time. The system mixes rapidly, so let us focus on the stationary behaviour. (In fact, it is because the system mixes rapidly that we are able to prove our concentration results.)

Theorem 1 ([5]) *Let d be a fixed positive integer, and suppose that the n-bin* *system is in the stationary distribution. Then there is an integer-valued function $m(n)$ such that aas the maximum load is $m(n)$ or $m(n)-1$: if $d=1$ then $m(n) = (1+o(1))\ln n/\ln\ln n$, and if $d \geq 2$ then $m(n) = \ln\ln n/\ln d + O(1)$.*

Now consider a second continuous-time model, the *supermarket model.* This is as before except that now bins are replaced by queues, each with a single unit-rate server, and $\lambda < 1$. There are similar results for this model.

Theorem 2 ([6]) *Let d be a fixed positive integer, and suppose that the n-queue system is in the stationary distribution. If $d = 1$, then aas the maximum queue length is within $\omega(n)$ of $\ln n/\ln(1/\lambda)$, where $\omega(n)$ is any function tending to ∞), and it is not concentrated on a bounded interval. If $d \geq 2$ then there is an integer valued function $m(n) = \ln\ln n/\ln d + O(1)$ such that aas the maximum queue length is $m(n)$ or $m(n)-1$.*

This is all joint work with Malwina Luczak. It arose from our endeavour to establish rigorous continuous-time results for routing in networks analogous to the discrete-time results in [7]. The 'bins' part of this work has recently been written up, the queues part nearly so: results on routing will follow later.

References

[1] Azar, Y., Broder, A., Karlin, A., Upfal, E. Balanced Allocations, *Proc. 26th ACM Symp. Theory Comp.* (1994) 593 – 602.

[2] Azar, Y., Broder, A., Karlin, A., Upfal, E. Balanced Allocations, *SIAM J. Comput.* **29** (1999) 180 – 200.

[3] Czumaj, A. Recovery time of dynamic allocation processes, *Proc. 10th Annual ACM Symp. on Parallel Algorithms and Architectures* (1998) 202 – 211.

[4] Graham, C. Kinetic Limits for Large Communication Networks, In *Modelling in Applied Sciences: A Kinetic Theory Approach*, N. Bellomo and M. Pulvirenti, eds., Birkhauser, 1999.

[5] Luczak, M.J. and McDiarmid, C. On the power of two choices: balls and bins in continuous time, preprint, 2003.

[6] Luczak, M.J. and McDiarmid, C. On the maximum queue length in the supermarket model, in preparation, 2004.

[7] Luczak, M.J., McDiarmid, C. and Upfal, E. On-line routing of random calls in networks, *Probab. Theory Relat. Fields.* **125** (2003) 457 – 482.

[8] Mitzenmacher, M., Richa, A.W. and Sitaraman, R. The power of two random choices: a survey of techniques and results, in preparation, 2002.

[9] Turner, S.R.E. The effect of increasing routeing choice on resource pooling, *Probability in Engineering and Informational Sciences* **12** (1998) 109 – 124.

[10] Vvedenskaya, N.D., Dobrushin, R.L., and Karpelevich, F.I. Queueing system with selection of the shortest of two queues: An asymptotic approach, *Problems of Information Transmission* **32** (1) (1996) 15–27.

Homomorphism Duality: On Short Answers to Exponentially Long Questions

Jaroslav Nešetřil

(joint work with Claude Tardif)

We give a new and more efficient construction of duals for general finite relational structures of a given type. We complement this by proving the superpolynomial lower bound for the size of the dual core. This bound is achieved even for duals of paths (i.e. for the type (2). This solves the main problem of [9].

Coloring problems belong to some of the central problems of combinatorics. Perhaps being encouraged by applications (such as channel assignement problems or Constraint Satisfaction type problems (CSP)) the recent revival of interest led to the investigation of many variants and far reaching generalizations, see e.g. [5, 3, 13]. The following problem captures both the difficulty and generality of some of this development:

H-coloring problem

Instance: A graph G;

Question: Does there exists a homomorphism $G \longrightarrow H$.

Recall, that a *homomorphism* $G \longrightarrow H$ is any mapping $f : V(G) \longrightarrow V(H)$ satisfying $f(x)f(y) \in E(H)$ whenever $xy \in E(G)$.

Thus for any complete graph $H = K_k$ the H-coloring problem reduces to the question whether the chromatic number $\chi(G)$ of graph G is $\leq k$. All CSP-problems may be expressed in a similar way as H-coloring problems for relational structures:

Let $\Delta = (\delta_i; i \in I)$ be a sequence of positive integers. A *relational structure of type* Δ (shortly Δ-structure) is a pair $(X, (R_i; i \in I))$ where X is a finite set and R_i is a δ_i-nary relation on X (i.e. we have $R_i \subset X^{\delta_i}$). Given a type Δ and

δ-systems $A = (X, (R_i; i \in I))$ and $A' = (X', (R'_i; i \in I))$ a *homomorphism* is a mapping $f : X \longrightarrow X'$ satisfying for every $i \in I$

$$(f(x_1), f(x_2), \dots, f(x_{\delta_i})) \in R'_i \text{ whenever } (x_1, x_2, \dots, x_{\delta_i}) \in R_i.$$

Given a structure H of type Δ we define the H-coloring problem in the complete analogy to graphs (yes, despite using for Δ-systems symbols A, B and the like, we still want to reserve H for the template of the coloring problem).

Viewing all this one expects that H-coloring problems are difficult to handle and that such problems tend to be computationally hard. This is indeed the case for undirected graphs. But for other types, and already for type (2) corresponding to the directed graphs, the situation is very difficult and there are many polynomial instances and the whole problem seems to be presently very difficult: there are many polynomial instances and even more hard cases, see e.g.[3, 2, 1].

This paper is devoted to the study of polynomial instances of H-coloring problems. Among those perhaps the simplest are those coloring problems which can be characterized by a simple obstruction set, by forbidden structures of a single type. This is expressed by the notion of the *(singleton) homomorphism duality*:

We say that a pair (F, H) of Δ-structures is a *dual pair* if the following equivalence holds for every Δ-structure A:

$$F \not\longrightarrow A \text{ iff } A \longrightarrow H.$$

The Δ-structure H is also called the *dual of* F and it is denoted by D_F. Note that up to homomorphism equivalence the dual D_F is uniquelly determined. One also sees easily that the only dual pair for undirected graphs (up to the homomorphism equivalence) is the pair (K_2, K_1), see [8] where this notion was first isolated. However one should not be discouraged by this as the richness of dualities lies in relational structures. Already for directed graphs (i.e. the type (2)) the duality pairs include pairs (P_k, T_k) where P_k is the monotone path of length k (i.e. with $k + 1$ vertices) and T_k is the transitive tournament with k vertices. One can see easily that these duality pairs correspond to the Hasse-Galai-Roy theorem: an undirected graph G has chromatic number $> k$ if and only if every orientation of G contains a monotone path of length k. Dualities represent a suprisingly rich scheme and many more dualities (and thus polynomial instances of coloring problems) were found [6, 7, 11]. Finally [9] characterize all homomorphism dualities (recall that a *core* of Δ structure A is the minimal structure which is homomorphism equivalent to A):

Theorem 1 *For every type Δ and for every Δ-tree T there exists a dual Δ-structure D_T. There are no other dual pairs.*

Viewing the difficulty of the classification of polynomial instances of H-coloring (already) for directed graphs it is perhaps surprising that one can achieve the full characterization of homomorphism dualities for general Δ-systems. The abundance of polynomial instances leads to the question about the nature of dual graphs. The proof given in [9] rests on some algebraic construction (such as the graph exponentiation) and on the reformulation of dualities in terms of homomorphism partial order $\mathcal{C}$ ("gaps" in $\mathcal{C}$). Thus dual structures D_T are complicated (and constructed indirectly) and their properties are non-trivial (and sometimes surprising, [10]). Thus it is desirable to have simpler explicite construction. Such a construction was provided in [11] for the case of directed graphs. This has been recently used in [12] to prove that the construction of the dual D_T is connected appart from isolated vertices.

In this paper we give a new construction of the dual for a general type Δ. This new construction is also more efficient: for a Δ-tree T with n vertices it produces the dual D_T of size $2^{n\log(n)}$ (as opposed to the double exponential bound which follows from [9]).

We complement this by providing examples which yield superpolynomial lower bound for cores of D_T. This improves the result of [9] and solves the main open problem left there. Perhaps surprisingly, in order to prove this lower bound we use relational structures (for large Δ).

The super polynomial lower bound for the size of core duals can be interpreted in the positive terms:

Corollary 2 *There are directed core graphs H such that $|V(H)| \geq 2^n$, and for every directed graph G, G is H-colorable if and only if every subgraph of G with at most $n\log(n)$ vertices is H-colorable.*

This an introduction to a paper by the same authors and the same title which is being submitted. It is available electronically at ITI Series and KAM-DIMATIA Series.

References

[1] A. Bulatov, A. Krokhin, P. Jeavons, Constraint Satisfaction Problems and Finite Algebras, Lecture Notes in Computer Science, vol. 1853, Springer Verlag 2000,p. 272–282.

[2] T. Feder, M. Vardi: The computational structure of monotone monadic SNP and constraint satisfaction: A study through datalog and group theory. SIAM J. of Computing 28 (1998), 57–104.

[3] P. Hell, J. Nešetřil, Graphs and Homomorphisms, Oxford University Press, 2003 (to appear).

[4] P. Hell, J. Nešetřil, X. Zhu, Duality and polynomial testing of tree homomorphisms. *Trans. Amer. Math. Soc.* **348** (1996), 1281–1297.

[5] T. Jensen, B. Toft: Graph Coloring Problems, Wiley-Interscience, New York , 1995.

[6] P. Komárek, Good characterisations in the class of oriented graphs (in czech), Ph. D. Thesis, Charles University, Prague, 1987.

[7] P. Komárek, Some new good characterizations for directed graphs. *Časopis Pěst. Mat.* **109** (1984), 348–354.

[8] J. Nešetřil, A. Pultr: On Classes of Relations and Graphs Determined by Subobjects and Factorobjects, Discrete Math. 22 (1978), 287–300.

[9] J. Nešetřil, C. Tardif, Duality Theorems for Finite Structures (Characterizing Gaps and Good Characterizations), *J. Combin. Theory Ser B* **80** (2000), 80–97.

[10] J. Nešetřil, C. Tardif, A dualistic approach to bounding the chromatic number of a graph, ITI Series 2001-036 (to appear).

[11] J. Nešetřil, C. Tardif, Density via Fuality, Theoretical Comp. Sci. 287, 2 (2002), 585–591.

[12] I. Švejdarová, Colouring of graphs and dual objects (in Czech), thesis, Charles University 2003.

[13] X. Zhu: Circular chromatic number: a survey, Discrete Math. 229 (2001), 371–410.

Extremal Connectivity for Topological Cliques

Deryk Osthus
(joint work with Daniela Kühn)

Given a natural number s, let $d(s)$ be the smallest number such that every graph of average degree $> d(s)$ contains a subdivision of the complete graph K_s of order s. The existence of $d(s)$ was proved by Mader [6]. As first observed by Jung [3], the complete bipartite graph $K_{t,t}$ with $t := \lfloor s^2/8 \rfloor$ shows that $d(s) \geq$

$\lfloor s^2/8 \rfloor$. Bollobás and Thomason [2] as well as Komlós and Szemerédi [4] showed that s^2 is the correct order of magnitude for $d(s)$. More precisely, it is known that

$$(1+o(1))\frac{9s^2}{64} \le d(s) \le (1+o(1))\frac{s^2}{2}. \tag{1}$$

The upper bound is due to Komlós and Szemerédi [4]. As observed by Łuczak, the lower bound is obtained by considering a random subgraph of a complete bipartite graph with edge probability 3/4. It is widely believed that the lower bound gives the correct constant, i.e. that random graphs provide the extremal graphs. If true, this would mean that the situation is similar as for ordinary minors. Indeed, Thomason [8] was recently able to prove that random graphs are extremal for minors and Myers [7] showed that all extremal graphs are essentially disjoint unions of pseudo-random graphs.

In [5] we showed that the lower bound in (1) is correct if we restrict our attention to bipartite graphs whose connectivity is close to their average degree:

Theorem 1 *Given $s \in \mathbb{N}$, let $c_{bip}(s)$ denote the smallest number such that every $c_{bip}(s)$-connected bipartite graph contains a subdivision of K_s. Then*

$$c_{bip}(s) = (1+o(1))\frac{9s^2}{64}.$$

In Theorem 1 the condition of being bipartite can be weakened to being H-free for some arbitrary but fixed 3-chromatic graph H. The proof of Theorem 1 builds on results and methods of Komlós and Szemerédi [4]. For arbitrary graphs, the best current upper bound on the extremal connectivity is the same as in (1). The proof of Theorem 1 yields the following improvement [5].

Theorem 2 *Given $s \in \mathbb{N}$, let $c(s)$ denote the smallest number such that every $c(s)$-connected graph contains a subdivision of K_s. Then*

$$(1+o(1))\frac{9s^2}{64} \le c(s) \le (1+o(1))\frac{s^2}{4}.$$

The lower bounds in Theorems 1 and 2 are provided by the random bipartite graphs mentioned above (since their connectivity is close to their average degree). Thus at least in the case of highly connected bipartite graphs they are indeed extremal.

By using methods as in the proof of Theorem 1, in [5] we also obtain a small improvement for the constant in the upper bound in (1).

Theorem 3 *Given $s \in \mathbb{N}$, let $d(s)$ denote the smallest number such that every graph of average degree $> d(s)$ contains a subdivision of K_s. Then*

$$(1+o(1))\frac{9s^2}{64} \le d(s) \le (1+o(1))\frac{10s^2}{23}.$$

The example of Łuczak mentioned above only gives us extremal graphs for Theorem 1 whose connectivity is about $3n/8$, i.e. whose connectivity is rather large compared to the order n of the graph. However, in [5] we showed that there are also extremal graphs whose order is arbitrarily large compared to their connectivity. In contrast to this, the situation for ordinary minors is quite different. In general a connectivity of order $s\sqrt{\log s}$ is needed to force a K_s minor, but the connectivity of the known extremal graphs is linear in their order. In fact, confirming a conjecture of Thomason [9], Böhme, Kawarabayashi and Mohar [1] proved that for all integers s there is an integer $n_0 = n_0(s)$ such that every graph of order at least n_0 and connectivity at least $45s$ contains the complete graph K_s as minor. Thus a linear connectivity suffices to force a K_s minor if we only consider sufficiently large graphs.

References

[1] T. Böhme, K. Kawarabayashi, B. Mohar, Linear connectivity forces dense minors, preprint 2003.

[2] B. Bollobás and A. Thomason, Proof of a conjecture of Mader, Erdős and Hajnal on topological complete subgraphs, *Eur. J. Comb.* **19** (1998), 883–887.

[3] H.A. Jung, Eine Verallgemeinerung des n-fachen Zusammenhangs für Graphen, *Math. Ann.* **187** (1970), 95–103.

[4] J. Komlós and E. Szemerédi, Topological cliques in graphs II, *Combin. Probab. Comput.* **5** (1996), 79–90.

[5] D. Kühn and D. Osthus, Extremal connectivity for topological cliques in bipartite graphs, submitted.

[6] W. Mader, Homomorphieeigenschaften und mittlere Kantendichte von Graphen, *Math. Annalen* **174** (1967), 265–268.

[7] J.S. Myers, Graphs without large complete minors are quasi-random, *Combin. Probab. Comput.*, to appear.

[8] A. Thomason, The extremal function for complete minors, *J. Combin. Theory B* **81** (2001), 318–338.

[9] A. Thomason, Extremal function for graph minors, preprint.

Constructions of Non-Principal Families in Extremal Hypergraph Theory

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(joint work with Dhruv Mubayi)

Here, we prove the non-principality phenomenon for the classical extremal problems for k-uniform hypergraphs. The main motivation is to study the qualitative difference between the cases $k = 2$, and $k \geq 3$, and our results for the Turán problem exhibit this difference.

Given a a family $\mathcal{F}$ of k-graphs, let $\mathrm{ex}(n, \mathcal{F})$ be the maximum size of an $\mathcal{F}$-free k-graph G on n vertices. Let $\pi(\mathcal{F})$ be the limit of $\mathrm{ex}(n, \mathcal{F})/\binom{n}{k}$ as $n \to \infty$. We call $\pi(\mathcal{F})$ the *Turán density* of $\mathcal{F}$.

Mubayi and Rödl [11] conjectured that there is a family $\mathcal{F}$ of 3-graphs such that

$$\pi(\mathcal{F}) < \min\{\pi(F) \mid F \in \mathcal{F}\}, \tag{1}$$

and commented that the result should even hold for a family $\mathcal{F}$ of size two. Balogh [1] proved the conjecture, calling this phenomenon the *non-principality* of the Turán function. This is in sharp contrast with the case of graphs ($k = 2$) where the Erdős-Stone-Simonovits Theorem [4, 2] applies.

However, Balogh's family has many graphs. Here we show how the so-called stability results lead to families $\mathcal{F}$ satisfying (1) and consisting of two k-graphs only. This approach succeeds for all even $k \geq 4$ and for $k = 3$, since it depends on stability results which are known only in these cases.

Non-Principal Families of Size 2

To obtain the *cone* $\mathrm{cn}(F)$ of a k-graph F, enlarge each edge of F by a new common vertex x:

$$\mathrm{cn}(F) := \{\{x\} \cup D \mid D \in F\}.$$

We call two order-n k-graphs F and G ε-*close* if we can make F isomorphic to G by adding and removing at most $\varepsilon\binom{n}{k}$ edges. A k-graph G is F-*extremal* if it is a maximum F-free k-graph of order $v(G)$. Let us call a k-graph F *stable* if any F-free k-graph G of order n with at least $(\pi(F) - o(1))\binom{n}{k}$ edges is $o(1)$-close to an F-extremal k-graph.

Lemma 1 *Let F be a stable k-graph. Suppose that we can find a k-graph H of order h such that $\pi(H) \geq \pi(F)$ and any F-extremal k-graph of order n contains*

$\Omega(n^h)$ *copies of* H*. Then*

$$\pi(\{F,H\}) < \min(\pi(F), \pi(H)). \tag{2}$$

Proof. Suppose on the contrary that $\pi(\{F,H\}) \geq \pi(F)$. Then there is an $\{F,H\}$-free k-graph G of order n and size $(\pi(F) - o(1))\binom{n}{k}$. Since F is stable, G is $o(1)$-close to an F-extremal k-graph G'. By hypothesis, G' contains $\Omega(n^h)$ copies of H. But each edge belongs to $O(n^{h-k})$ H-subgraphs, so we cannot destroy all of them by removing $o(n^k)$ edges. This is a contradiction to $G \not\subset H$. ∎

Theorem 2 *For even* $k \geq 4$ *and for* $k = 3$ *there are* k*-graphs* F *and* H *satisfying (2).*

Proof. Let $k = 2l$ be even. Let $F = \{A \cup B, A \cup C, B \cup C\}$, where A, B, C are disjoint l-sets. Frankl [5] showed that $\pi(F) = \frac{1}{2}$. Keevash and Sudakov [9, Theorem 3.4] showed that F is stable. Every extremal k-graph G' for F on $n \geq n_0$ vertices has vertex partition $X \cup Y$, $|X| \approx |Y| \approx \frac{n}{2}$, and consists of all edges intersecting X (and also Y) in an odd number of vertices.

Let us take $H = \mathrm{cn}(K_m^{k-1})$ where $m = m(k)$ is a sufficiently large integer to satisfy $\frac{k!}{m^k}\binom{m}{k} > \frac{1}{2}$. The latter implies that $\pi(H) > \frac{1}{2}$, because the blown-up K_m^k does not contain H. As G' contains $(2 + o(1))\,\frac{n}{2}\binom{n/2}{m}$ copies of H, Lemma 1 implies that the family $\{F, H\}$ has the required properties.

For $k = 3$ we can use the stability result either for the Fano plane, (established independently by Füredi and Simonovits [7] and by Keevash and Sudakov [8]), or for $F_{3,2}$, established by Füredi, Pikhurko, and Simonovits [6]. In both cases we can take $H = \mathrm{cn}(K_m^2)$ for some sufficiently large m. ∎

Concluding Remarks

For the case of odd $k \geq 5$, we can build upon the ideas in [1] and construct a non-principal k-graph family $\mathcal{F}$ for every $k \geq 3$, see [10]. The obtained family consists of finitely many k-graphs; however, this approach does not seem to give $|\mathcal{F}| = 2$.

One can also consider the *Ramsey-Turán* density $\rho(\mathcal{F})$ where in addition to being $\mathcal{F}$-free we require that the maximum size of an independent set of G is $o(n)$. (This problem was introduced by Erdős and Sós [3].) One can show that for $k \geq 3$ if $\mathcal{F}$ is a non-principal k-graph family with respect to the Turán density then $\mathcal{F}(2)$ is non-principal with respect to the Ramsey-Turán density, see [10]. Here $\mathcal{F}(2)$ is obtained by blowing-up each member of $\mathcal{F}$ by factor of 2.

Curiously, the situation with graphs remains open.

Problem 3 *Do there exist* 2*-graphs* G_1, G_2 *for which*

$$\rho(\{G_1, G_2\}) < \min\{\rho(G_1), \rho(G_2)\}?$$

What about if we require $\rho(\{G_1, G_2\}) > 0$ *as well?*

References

[1] J. Balogh. The Turán density of triple systems is not principal. *J. Combin. Theory* (A), 100:176–180, 2002.

[2] P. Erdős and M. Simonovits. A limit theorem in graph theory. *Stud. Sci. Math. Hungar.*, pages 51–57, 1966.

[3] P. Erdős and V. T. Sós. On Ramsey-Turán type theorems for hypergraphs. *Combinatorica*, 2:289–295, 1982.

[4] P. Erdős and A. H. Stone. On the structure of linear graphs. *Bull. Amer. Math. Soc.*, 52:1087–1091, 1946.

[5] P. Frankl. Asymptotic solution of a Turán-type problem. *Graphs Combin.*, 6:223–227, 1990.

[6] Z. Füredi, O. Pikhurko, and M. Simonovits. Triple systems with empty neighborhoods. Submitted, 2004.

[7] Z. Füredi and M. Simonovits. Triple systems not containing a Fano configuration. Submitted, 2003.

[8] P. Keevash and B. Sudakov. The exact Turán number of the Fano plane. Submitted, 2003.

[9] P. Keevash and B. Sudakov. On a hypergraph Turán problem of Frankl. Submitted, 2003.

[10] D. Mubayi and O. Pikhurko. Constructions of non-principal families in extremal hypergraph theory. Submitted, 2003.

[11] D. Mubayi and V. Rödl. On the Turán number of triple systems. *J. Combin. Theory* (A), 100:135–152, 2002.

The Phase Transition in the Uniformly Grown Random Graph has Infinite Order

Oliver Riordan
(joint work with Béla Bollobás and Svante Janson)

The emergence of a giant component is one of the most frequently studied phenomena in the theory of random graphs. Much of the interest is due to the fact that a giant component in a finite graph corresponds to an infinite component, or 'infinite cluster', in percolation on an infinite graph. In fact, it can be argued that it is more important and more difficult to study detailed properties of the emergence of the giant component than to study the corresponding infinite percolation near the critical probability.

The quintessential example of the emergence of a giant component is in the classical random graph model $G_{n,p}$, the graph with vertex set $\{1, 2, \dots, n\}$ in which each pair of vertices is joined with probability p, independently of all other pairs. Let us say that an event holds *with high probability* (**whp**), if it holds with probability tending to 1 as $n \to \infty$. In 1960, Erdős and Rényi [7, 8] showed that the critical probability for $G_{n,p}$ is $1/n$: if $c < 1$ is a constant then **whp** the largest component of $G_{n,c/n}$ has $O(\log n)$ vertices, while there is a function $\theta(c) > 0$ such that for constant $c > 1$, **whp** $G_{n,c/n}$ has a component of order $(\theta(c) + o(1))n$, and no other component of order larger than $O(\log n)$. The proper 'window' of the phase transition was found much later by Bollobás [1] and Łuczak [10]. In $G_{n,c/n}$ the giant component emerges rather rapidly: the right-derivative of $\theta(c)$ at $c = 1$ is 2; this makes the study of the phenomenon manageable.

Our task here is considerably harder, since in the model we shall study the giant component emerges much more slowly. Our model, $G_n(c)$, is the finite version of a model first proposed by Dubins in 1984 (see [9, 11]): it is parametrized by n, the number of vertices, and a constant $c > 0$ to which edge probabilities are proportional, just as for $G_{n,c/n}$. It can be read out of results of Kalikow and Weiss [9] and Shepp [11] that there is a critical value $c = 1/4$ above which a giant component is present. In $G_n(c)$, the transition from having no giant component to having a giant component is rather tantalizing, since it is very slow indeed. It turns out that for any c less than $1/4$, **whp** the largest component of $G_n(c)$ already contains $n^{\Theta(1)}$ vertices, which is *much* larger than the $O(\log n)$ we have in $G_{n,a/n}$, $a < 1$. For $c > 1/4$, **whp** there is a giant component of order proportional to n, and the other components are small. In fact, there is a function $\phi(c)$, equal to 0 for $c \leq 1/4$ but positive for $c > 1/4$, such that **whp** the largest component of $G_n(c)$ has order $(\phi(c) + o(1))n$. However, rather than having positive right-derivative at the critical point, in this case (if the derivatives exist) every derivative of $\phi(c)$ at $c = 1/4$ is zero. This phenomenon is often called a phase transition of *infinite* *order*. Somewhat surprisingly, in spite of this extremely gentle growth of the giant component, we can give good bounds on $\phi(c)$ from above and below, showing, in particular, that $\phi(1/4+\epsilon) = o(\epsilon^k)$ for every k.

A somewhat similar, although less surprising, phenomenon was studied in [2], where for a different model it was shown that for *every* positive value of the appropriate parameter c there is a giant component, but its normalized size has all derivatives zero at $c = 0$. Nevertheless, a gentle increase at the very beginning is considerably less suprising than a 'sudden' gentle increase in a function which is zero up to some positive value.

Turning to the model, in [3], Callaway, Hopcroft, Kleinberg, Newman and Strogatz introduced a simple new model (which we shall call the CHKNS model) for random graphs growing in time. They gave heuristic arguments to find the critical point for the percolation phase transition in this graph, and numerical results (from integrating an equation, rather than just simulating the graph) to suggest that this transition has infinite order. Heuristic arguments for an infinite order phase transition in this and other models have been given by Dorogovtsev, Mendes and Samukhin [4].

Here we consider an even simpler and more natural model, the *uniformly grown random graph*, or '*$1/j$-graph*'. This is the finite version of a model proposed by Dubins in 1984. We define the $1/j$-graph $G_n^{1/j}$ as the random graph on $\{1, 2, \dots, n\}$ in which each pair $i < j$ of vertices is joined independently with probability $1/j$. We may think of $G_n = G_n^{1/j}$ as a graph growing in time, where each vertex joins to a set of earlier vertices chosen uniformly at random, the set itself having a random size, which is essentially Poisson with mean 1. We study the random subgraph $G_n(c)$ of G_n obtained by selecting edges independently with probability $c < 1$. Of course, $G_n(c)$ can be defined directly by specifying that each pair $i < j$ is joined independently with probability c/j. With this definition, values of c greater than one make sense, provided we replace c/j by $\max\{c/j, 1\}$.

Kalikow and Weiss [9] showed that for $c < 1/4$ the infinite version $G_\infty(c)$ of $G_n(c)$ is disconnected with probability one. It is implicit in their work that **whp** the largest component in the finite graph $G_n(c)$, $c < 1/4$, has order $o(n)$. In the other direction, Shepp [11] showed that for $c > 1/4$, $G_\infty(c)$ is connected with probability 1; his proof involved showing that $G_n(c)$ has a component of order $\Theta(n)$ with probability bounded away from zero. Hence, the threshold for the emergence of a giant component in $G_n(c)$ is at $c = 1/4$. A similar result for a considerably more general model was proved by Durrett and Kesten [6].

Here we study the size of the giant component above the threshold, showing that the giant component emerges very slowly.

Theorem 1 *There is a function $\phi(c)$ such that as $n \to \infty$ with $c \geq 0$ fixed,* **whp** *the largest component of $G_n^{1/j}(c)$ contains $(\phi(c) + o(1))n$ vertices.*

Furthermore, $\phi(c) = 0$ *for* $c \leq 1/4$*, and*

$$\phi(c) = \exp\left(-\frac{\pi + o(1)}{2\sqrt{c - 1/4}}\right)$$

as c *tends to* $1/4$ *from above.*

In particular, $\phi(1/4 + \epsilon) = o(\epsilon^k)$ for any k, and the phase transition is of 'infinite order'.

Although we work with the $1/j$-graph, as it has a simpler and more natural static description, all our results carry over to the CHKNS model. As pointed out independently by Durrett [5], this is also true of the earlier threshold results, which predate the CHKNS model by 10 years!

References

[1] B. Bollobás, The evolution of random graphs, *Trans. Amer. Math. Soc.* **286**, 257–274 (1984).

[2] B. Bollobás and O. Riordan, Robustness and vulnerability of scale-free random graphs, *Internet Mathematics* **1** (2003), 1–35.

[3] D.S. Callaway, J.E. Hopcroft, J.M. Kleinberg, M.E.J. Newman and S.H. Strogatz, Are randomly grown graphs really random?, *Phys. Rev. E* **64** (2001), 041902.

[4] S.N. Dorogovtsev, J.F.F. Mendes and A.N. Samukhin, Anomalous percolation properties of growing networks, *Phys. Rev. E* **64** (2001), 066110.

[5] R. Durrett, Rigorous result for the CHKNS random graph model, Proceedings, Discrete Random Walks 2003, Cyril Banderier and Christian Krattenthaler, Eds. *Discrete Mathematics and Theoretical Computer Science* **AC** (2003), 95–104. `http://dmtcs.loria.fr/proceedings/`

[6] R. Durrett and H. Kesten, The critical parameter for connectedness of some random graphs, in *A tribute to Paul Erdős*, Cambridge Univ. Press, Cambridge, 1990, pp 161–176,

[7] P. Erdős and A. Rényi, On the evolution of random graphs, *Magyar Tud. Akad. Mat. Kutató Int. Közl.* **5**, 17–61 (1960).

[8] P. Erdős and A. Rényi, On the evolution of random graphs, *Bull. Inst. Internat. Statist.* **38**, 343–347, (1961).

[9] S. Kalikow and B. Weiss, When are random graphs connected?, *Israel J. Math.* **62** (1988), 257–268.

[10] T. Łuczak, Component behavior near the critical point of the random graph process, *Random Structures and Algorithms* **1**, 287–310 (1990).

[11] L.A. Shepp, Connectedness of certain random graphs. *Israel J. Math.* **67** (1989), 23–33.

The Regularity Method for k-uniform Hypergraphs

Vojtěch Rödl

(joint work with Brendan Nagle, Mathias Schacht and Jozef Skokan)

The Regularity Lemma of Szemerédi [20], proved to be a powerful tool in Combinatorics. This lemma states that all sufficiently large graphs can be approximated, in some sense, by random graphs. Since "random-like" graphs are often easier to handle than arbitrary graphs, the Regularity Lemma is especially useful in situations when the problem in question is easier to prove for random graphs.

Let $G = (V, E)$ be a graph and A, $B \subseteq V$ be a pair of disjoint sets of vertices of G. Denote by $e(A, B)$ the number of edges of G between A and B. The *density* of the pair (A, B) is defined by $d(A, B) = e(A, B)/(|A||B|)$. The pair is called ε-*regular* if for any $A' \subseteq A$, $B' \subseteq B$ with $|A'| \geq \varepsilon|A|$, $|B'| \geq \varepsilon|B|$, we have $|d(A, B) - d(A', B')| < \varepsilon$.

Theorem 1 (Szemerédi's Regularity Lemma) *For every $\varepsilon > 0$ there exist a T_0 such that the vertex set $V(G)$ of any graph G can be partitioned into $t \leq T_0$ classes $V(G) = V_1 \cup \cdots \cup V_t$, so that all but εt^2 pairs (V_i, V_j) are ε-regular.*

Many applications of the Regularity Lemma are based on its accompanying Counting Lemma (see, e.g., [9, 10] for a survey).

Theorem 2 (Counting Lemma) *If G is an ℓ-partite graph with $V(G) = V_1 \cup \cdots \cup V_\ell$ and $|V_i| = n$ for all $i \in [\ell]$, and all pairs (V_i, V_j) are ε-regular of density d for $1 \leq i < j \leq \ell$, then G contains $(1 \pm f_\ell(\varepsilon))d^{\binom{\ell}{2}} \times n^\ell$ cliques K_ℓ of order ℓ, where $f_\ell(\varepsilon) \to 0$ as $\varepsilon \to 0$.*

We discuss a generalization of Szemerédi's Regularity Lemma from graphs to k-uniform hypergraphs, which allows us to prove an accompanying Counting Lemma.

Unlike for graphs, there are several "natural ways" to define "regularity" for k-uniform hypergraphs. Consequently, various forms of a Regularity Lemma for hypergraphs have been already considered in [1, 2, 4, 6, 13]. None of these Regularity Lemmas seemed to admit a companion counting result (i.e., a corresponding generalization of Theorem 2). The first attempt of developing a Hypergraph Regularity Lemma with a corresponding Counting Lemma was undertaken by Frankl and the speaker in [5] for 3-uniform hypergraphs. Recently, the speaker in collaboration with Skokan [17] established a generalization of this Regularity Lemma to k-uniform hypergraphs for any $k \geq 3$.

Analogously to the feature that Szemerédi's Regularity Lemma decomposes a given graph into an ε-regular partition, this Hypergraph Regularity Lemma decomposes the edge set of a given k-uniform hypergraph into constantly many "blocks", almost all of which are, in a specific sense, "quasi-random". The concept of hypergraph regularity which plays the analogous role of the ε-regular pair is, unfortunately, considerably more technical than its graph counterpart, and we cannot give the precise definitions here.

Just as Theorem 2, the Counting Lemma, is an important companion statement to Szemerédi's Regularity Lemma, most applications of the Hypergraph Regularity Lemma from [17] require a similar companion lemma - the "general Counting Lemma". Analogously to Theorem 2, the general Counting Lemma estimates the number of copies of the clique $K_\ell^{(k)}$ (i.e., the complete k-uniform hypergraph on ℓ vertices) contained in an appropriate collection of "dense and regular blocks" within a regular partition provided by the Hypergraph Regularity Lemma. Such a Counting Lemma was established for special cases ($k = 3$, $\ell > 3$ and $k = 4$, $\ell = 5$) in [5, 11, 16]. Recently, in [12] Nagle, Schacht and the speaker, succeeded to prove the general Counting Lemma for any $\ell > k \geq 2$, reducing it to an earlier result from [8]. This Counting Lemma together with the Hyergraph Regularity Lemma of [17] can be viewed as a generalization of the Regularity Method from graphs to uniform hypergraphs. A similar extension was independently obtained by Gowers [7].

These generalizations can be applied to several extremal hypergraph problems. In particular, answering a question of Erdős, Frankl, and speaker [3], we proved the following theorem in [15]

Theorem 3 *Suppose an n-vertex k-uniform hypergraph $\mathcal{H}$ contains only $o(n^\ell)$ copies of $K_\ell^{(k)}$. Then one can delete $o(n^k)$ edges of $\mathcal{H}$ to make it $K_\ell^{(k)}$-free.*

It is known that this theorem can be used to give an alternative proof the well-known Density Theorem of Szemerédi regarding the upper density of sets containing no arithmetic progression of fixed length (see [5, 15]). Moreover, it can also be used to derive combinatorial proofs to some of the density theorems of Furstenberg and Katznelson (see [7, 14, 18]).

References

[1] F. R. K. Chung, *Regularity lemmas for hypergraphs and quasi-randomness*, Random Structures Algorithms **2** (1991), no. 2, 241–252.

[2] A. Czygrinow and V. Rödl, *An algorithmic regularity lemma for hypergraphs*, SIAM J. Comput. **30** (2000), no. 4, 1041–1066 (electronic).

[3] P. Erdős, P. Frankl, and V. Rödl, *The asymptotic number of graphs not containing a fixed subgraph and a problem for hypergraphs having no exponent*, Graphs Combin. **2** (1986), no. 2, 113–121.

[4] P. Frankl and V. Rödl, *The uniformity lemma for hypergraphs*, Graphs Combin. **8** (1992), no. 4, 309–312.

[5] ———, *Extremal problems on set systems*, Random Structures Algorithms **20** (2002), no. 2, 131–164.

[6] A. Frieze and R. Kannan, *Quick approximation to matrices and applications*, Combinatorica **19** (1999), no. 2, 175–220.

[7] W. T. Gowers, *Hypergraph regularity and the multidimensional Szemerédi theorem*, manuscript.

[8] Y. Kohayakawa, V. Rödl, and J. Skokan, *Hypergraphs, quasi-randomness, and conditions for regularity*, J. Combin. Theory Ser. A **97** (2002), no. 2, 307–352.

[9] J. Komlós, A. Shokoufandeh, M. Simonovits, and E. Szemerédi, *The regularity lemma and its applications in graph theory*, Theoretical aspects of computer science (Tehran, 2000), Lecture Notes in Comput. Sci., vol. 2292, Springer, Berlin, 2002, pp. 84–112.

[10] J. Komlós and M. Simonovits, *Szemerédi's regularity lemma and its applications in graph theory*, Combinatorics, Paul Erdős is eighty, Vol. 2 (Keszthely, 1993), Bolyai Soc. Math. Stud., vol. 2, János Bolyai Math. Soc., Budapest, 1996, pp. 295–352.

[11] B. Nagle and V. Rödl, *Regularity properties for triple systems*, Random Structures Algorithms **23** (2003), no. 3, 264–332.

[12] B. Nagel, V. Rödl, and M. Schacht, *The counting lemma for regular k-uniform hypergraphs*, manuscript.

[13] H. J. Prömel and A. Steger, *Excluding induced subgraphs. III. A general asymptotic*, Random Structures Algorithms **3** (1992), no. 1, 19–31.

[14] V. Rödl, M. Schacht, E. Tengan, and N. Tokushige, *Density theorems and extremal hypergraph problems*, manuscript.

[15] V. Rödl and J. Skokan, *Applications of the regularity lemma for uniform hypergraphs*, manuscript.

[16] ———, *Counting subgraphs in quasi-random 4-uniform hypergraphs*, submitted.

[17] ———, *Regularity lemma for k-uniform hypergraphs*, to appear in Random Structures Algorithms.

[18] J. Solymosi, *Note on a question of Erdős and Graham*, to appear in Combin. Probab. Comput.

[19] E. Szemerédi, *On sets of integers containing no k elements in arithmetic progression*, Acta Arith. **27** (1975), 199–245, Collection of articles in memory of Juriĭ Vladimirovič Linnik.

[20] ———, *Regular partitions of graphs*, Problèmes combinatoires et théorie des graphes (Colloq. Internat. CNRS, Univ. Orsay, Orsay, 1976), CNRS, Paris, 1978, pp. 399–401.

Graph Parameters and Reflection Positivity

Alexander Schrijver

(joint work with Michael H. Freedman and László Lovász [1])

We characterize which real-valued (undirected) graph parameters are of the following type, where H is a graph and $\alpha : VH \to \mathbb{R}_+$ and $\beta : EH \to \mathbb{R}$:

$$f_{H,\alpha,\beta}(G) := \sum_{\substack{\phi:VG\to VH \\ \phi \text{ homomorphism}}} \Big(\prod_{v\in VG} \alpha_{\phi(v)}\Big)\Big(\prod_{uv\in EG} \beta_{\phi(u)\phi(v)}\Big). \tag{1}$$

Here $\phi : VG \to VH$ is a *homomorphism* if $\phi(u)\phi(v) \in EH$ for all $uv \in EG$. (So if $\phi(u) = \phi(v)$, then H has a loop at $\phi(u)$.) To reduce technicalities, it has turned out to be convenient to assume that G has no loops but may have multiple edges, while H has no multiple edges but may have loops.

Several graph parameters are indeed of this type. A first example of such a parameter is $f(G) :=$ the number of k-vertex-colourings of G (for some fixed k). Then we can take $H = K_k$ (the complete loopless graph on k vertices), and α and β the all-one functions on VG and EG respectively. More generally, by taking any graph H and $\alpha \equiv 1$ and $\beta \equiv 1$, $f(G)$ counts the number of homomorphism of G into H. By taking H to be a two-vertex graph with one edge connecting the two vertices and a loop at one of the two vertices, $f(G)$ then counts the number of stable sets of G.

Other examples are given by the partition functions of several models in statistical mechanics. Then H can be taken to be a complete graph with all loops attached, and VH is interpreted as the set of states certain elements of a system G can adopt. The function $\beta : EH \to \mathbb{R}$ describes the energy of the interaction of two neighbouring states, while $\alpha : VH \to \mathbb{R}_+$ can be the external energy of the different states, or, alternatively, if $\sum_{v \in VH} \alpha_v = 1$, α_v may be the probability that an element is in state v. Then any function $\phi : VG \to VH$ is a *configuration* of system G, and $f_{H,\alpha,\beta}(G)$ is the total or average energy of the system. (A different interpretation of this model is in economics, where β gives the profit or cost of certain interactions, and $f_{H,\alpha,\beta}$ gives the expected profit or cost.)

It will follow from our theorem (but also a direct construction based on characters can be made) that also the following graph parameters are of the type above. Let Γ be a finite abelian group and let S be a subset of Γ with $-S = S$ (i.e., $-s \in S$ if $s \in S$). For any graph G, fix an arbitrary orientation. Call a function $x : EG \to \Gamma$ an *S-flow* if all values of x are in S and x satisfies the flow conservation law at each vertex v of G: the inflow is equal to the outflow. Let $f(G)$ be the number of S-flows. (Since $-S = S$, this number is independent of the orientation chosen.) A well-known example is when Γ is the cyclic group with k elements and $S = \Gamma \setminus \{0\}$. Then an S-flow corresponds to a nowhere-zero k-flow, and Tutte's nowhere-zero 5-flow conjecture says that $f(G) > 0$ if $k = 5$ and G has no bridges. (It can be shown that for the case of nowhere-zero k-flows, we can take for H the complete graph on k vertices with all loops attached, and set $\alpha(v) = 1/k$ for each $v \in VH$, $\beta(e) = k - 1$ for each nonloop edge e of H, and $\beta(e) = -1$ for each loop e of H.)

The question of characterizing the graph parameters of form (1) is motivated, among others, by the question of the physical realizability of certain graph parameters. It turns out that two conditions on certain matrices derived from the graph parameter are necessary and sufficient: restricted (namely exponential) growth of the ranks and positive semidefiniteness — a condition that corresponds to the well-known *reflection positivity* in statistical mechanics.

These matrices are described as follows. For any integer $k \geq 0$, let $\mathcal{G}_k$ be the set of graphs in which k of the vertices are labeled $1, \ldots, k$, while the remaining vertices are unlabeled. For $G, G' \in \mathcal{G}_k$, let GG' denote the graph obtained by first taking the disjoint sum of G and G', and next identifying equally labeled vertices. (So GG' has $|VG| + |VG'| - k$ vertices.) For any graph parameter f, let $M_{f,k}$ be the (infinite) $\mathcal{G}_k \times \mathcal{G}_k$ matrix whose entry in position G, G' is equal to $f(GG')$.

Then for any graph parameter f (where K_0 is the graph with no vertices and edges):

Theorem 1 *There exist H, $\alpha : VH \to \mathbb{R}_+$ and $\beta : EH \to \mathbb{R}$ such that $f = f_{H,\alpha,\beta}$ if and only if $f(K_0) = 1$ and there exists a c such that each $M_{f,k}$ is positive semidefinite and has degree at most c^k.*

Necessity can be shown rather straightforwardly. The method for proving sufficiency is based on considering each $\mathcal{G}_k$ as a semigroup (taking GG' above as multiplication), making the semigroup algebra over $\mathcal{G}_k$, and taking the quotient algebra over the null-space of $M_{f,k}$, thus obtaining a finite-dimensional Banach algebra, which has a basis of idempotents. The interaction of the idempotents between these algebras for different values of k gives us the combinatorics to find H and the functions α and β.

Extension of this method gives similar results for directed graph and hypergraph parameters, and more generally for any parameter for systems that have a certain semigroup structure.

References

[1] M.H. Freedman, L. Lovász, A. Schrijver, Reflection positivity, rank connectivity, and homomorphisms of graphs, preprint, 2003.

Claw-free Graphs

Paul Seymour

(joint work with Maria Chudnovsky)

A graph is *claw-free* if no induced subgraph is isomorphic to the complete bipartite graph $K_{1,3}$. We give a structural description of all claw-free graphs with the additional property that every vertex is in a 3-vertex stable set.

One way to formulate our result is that, for every claw-free graph G, either G belongs to one of (about ten) well-understood basic classes of graphs, or G admits one of (about five) types of decomposition, or some vertex is not in a stable set of size 3. Having proved that, we can stand back and ask, what does this tell us about the "global structure" of G? And there is indeed a "structure theorem", but we are still working on its precise formulation, and for this abstract we confine ourselves to the decomposition theorem.

First, here are a few kinds of claw-free graphs.

- **Line graphs.** If H is a graph, its *line graph* $L(H)$ is the graph with vertex set $E(H)$, in which distinct $e, f \in E(H)$ are adjacent if and only if they have a common end in H.
- **The icosahedron.** This is the unique planar graph with twelve vertices all of degree five.
- **The Schläfli graph.** Let G be the graph with 27 vertices $a_{i,j,k}$ ($1 \leq i, j, k \leq 3$), and with adjacency as follows. $a_{i,j,k}$ is adjacent to $a_{i',j',k'}$ if and only if either

- $k' = k$ and either $i' = i$ or $j' = j$, or
- $k' = k + 1 \pmod 3$ and $j' \neq i$, or
- $k' = k + 2 \pmod 3$ and $i' \neq j$.

- **Circular interval graphs.** Let Σ be a circle and let $F_1, \ldots, F_k$ be subsets of Σ, each homeomorphic to the closed interval $[0, 1]$, and no three with union Σ. Let V be a finite subset of Σ, and let G be the graph with vertex set V in which $v_1, v_2 \in V$ are adjacent if and only $v_1, v_2 \in F_i$ for some i.

- **XX-configurations.** Let G be the graph with vertex set $\{v_1, \ldots, v_{13}\}$, with adjacency as follows. v_1-$\cdots$-v_6 is a hole in G of length 6. Next, v_7 is adjacent to v_1, v_2; v_8 is adjacent to v_4, v_5, and possibly to v_7; v_9 is adjacent to v_6, v_1, v_2, v_3; v_{10} is adjacent to v_3, v_4, v_5, v_6, v_9; v_{11} is adjacent to $v_3, v_4, v_6, v_1, v_9, v_{10}$; v_{12} is adjacent to $v_2, v_3, v_5, v_6, v_9, v_{10}$; and v_{13} is adjacent to $v_1, v_2, v_4, v_5, v_7, v_8$.

- **An extension of $L(K_6)$.** Let H be the graph with seven vertices $h_0, \ldots, h_6$, in which $h_1, \ldots, h_6$ are pairwise adjacent and h_0 is adjacent to h_1. Let G be the graph obtained from the line graph $L(H)$ of H by adding one new vertex, adjacent precisely to the members of $V(L(H)) = E(H)$ that are not incident with h_1 in H.

- **The graph of crosses.** Let $k \geq 1$. Let G have vertex set the union of nine disjoint sets $A_{i,j}$ $(1 \leq i, j \leq 3)$, where $A_{2,1}$,$A_{2,3}$, $A_{1,2}$,$A_{3,2}$ all have cardinality k, and the other five have cardinality 1. Let every vertex of $A_{i,j}$ be adjacent to every vertex of $A_{i',j'}$ if either $i = i'$ or $j = j'$, and otherwise let there be no edges between $A_{i,j}$ and $A_{i',j'}$. Now we need to change the adjacency between the four sets $A_{2,1}, A_{2,3}, A_{1,2}, A_{3,2}$. Order each of these four sets. If u is the ith vertex of one of these four sets, say $A_{a,b}$, and v is the jth vertex of another of these sets, say $A_{c,d}$, let u, v be adjacent if either

  - $i = j$ and $a \neq c$ and $b \neq d$, or
  - $i \neq j$ and either $a = c$ or $b = d$.

- **The path of triangles.** Let G have vertices $v_1, \ldots, v_n$ with n odd, in which for $i < j$, v_i is adjacent to v_j if either $j - i = 1$, or $j - i = 2$ and i is odd, or $j - i \geq 3$ and $j - i = 2 \bmod 3$.

For each of these types of graph, we regard the graphs of that type and all their induced subgraphs as forming one of our basic classes. These are the nicest of our classes; there are a few others, quite similar, that we omit. (We shall not attempt a precise statement of the theorem here.)

Next, decompositions. Two subsets X, Y of $V(G)$ with $X \cap Y = \emptyset$ are *complete* to each other if every vertex of X is adjacent to every vertex of Y, and *anticomplete* if no vertex in X is adjacent to a member of Y.

Distinct vertices u, v of G are *twins* (in G) if they are adjacent and have exactly the same neighbours in $V(G) \setminus \{u, v\}$. Admitting twins is the first of our decompositions.

Now let A, B be disjoint subsets of $V(G)$. The pair (A, B) is called a *homogeneous pair of cliques* if

- A, B are both cliques
- every vertex $v \in V(G) \setminus (A \cup B)$ is either A-complete or A-anticomplete and either B-complete or B-anticomplete, and
- A is neither complete nor anticomplete to B.

The third kind of decomposition is a *1-join*. Suppose that V_1, V_2 partition $V(G)$, and for $i = 1, 2$ there is a subset $A_i \subseteq V_i$ such that:

- for $i = 1, 2$, A_i is a clique, and $A_i, V_i \setminus A_i$ are both nonempty
- A_1 is complete to A_2
- every edge between V_1 and V_2 is between A_1 and A_2.

In these circumstances, we say that (V_1, V_2) is a *1-join*.

Next, suppose that V_0, V_1, V_2 are disjoint subsets with union $V(G)$, and for $i = 1, 2$ there are subsets A_i, B_i of V_i satisfying the following:

- for $i = 1, 2$, A_i, B_i are cliques, $A_i \cap B_i = \emptyset$ and A_i, B_i and $V_i \setminus (A_i \cup B_i)$ are all nonempty
- A_1 is complete to A_2, and B_1 is complete to B_2, and there are no other edges between V_1 and V_2, and
- V_0 is a clique; and for $i = 1, 2$, V_0 is complete to $A_i \cup B_i$ and anticomplete to $V_i \setminus (A_i \cup B_i)$.

We call the triple (V_1, V_0, V_2) a *2-join*. (This is closely related to, but not quite the same as, what has been called a 2-join in other papers.)

The fifth and last decomposition is the following. Let (V_1, V_2) be a partition of $V(G)$, such that for $i = 1, 2$ there are cliques $A_i, B_i, C_i \subseteq V_i$ with the following properties:

- For $i = 1, 2$ the sets A_i, B_i, C_i are pairwise disjoint and have union V_i

- V_1 is complete to V_2 except that there are no edges between A_1 and A_2, between B_1 and B_2, and between C_1 and C_2.
- V_1, V_2 are both nonempty.

In these circumstances we say that G is a *hex-join* of $G|V_1$ and $G|V_2$. Note that if G is expressible as a hex-join as above, then the sets $A_1 \cup B_2, B_1 \cup C_2$ and $C_1 \cup A_2$ are three cliques with union $V(G)$, and consequently no graph G with $\alpha(G) > 3$ admits a hex-join. ($\alpha(G)$ denotes the size of the largest stable set in G.)

Let us say a *triad* in G is a stable set of vertices with cardinality 3. Our main theorem, then, says:

Theorem 1 *For every connected claw-free graph in which every vertex belongs to a triad, either G belongs to one of the basic classes, or G admits either twins, a homogeneous pair of cliques, a 1-join, a 2-join or a hex-join.*

It is convenient to break the proof (and indeed, the full statement of the theorem) into four cases:

- $\alpha(G) \geq 4$
- $\alpha(G) \leq 3$, but there are four vertices so that only one pair of them is adjacent
- for every triad, every vertex not in X has exactly two neighbours in X, and every vertex is in a triad
- for every triad, every vertex not in X has exactly two neighbours in X, and some vertex is not in any triad.

In each case (except the fourth, where we have nothing to say), we have a result that "either G belongs to a basic class or G admits a decomposition", but the basic classes and decompositions are different for different types. We omit further details here. Some of these results are written in [1, 2].

References

[1] Maria Chudnovsky and Paul Seymour, "Claw-free graphs. II. Circular interval graphs", manuscript, October 2003.

[2] Maria Chudnovsky and Paul Seymour, "Claw-free graphs. III. Sparse decomposition", manuscript, October 2003.

Paradoxical Decompositions and Growth Properties

Vera T. Sós

The theory of paradoxical decompositions arose in connection with the existence of non-Lebesgue measurable sets.

The non-existence of isometry-invariant finitely additive measure in $\mathbb{R}^3$ was proved by Banach and Tarski (1924) [1] by means of paradoxical decomposition. They proved that it is possible to partition the unit ball in $\mathbb{R}^3$ into finitely many pieces and to rearrange them by rigid motions (using isometric transformations) to form two unit balls. This "duplication", this "paradoxical decomposition" of the ball at first seems to be impossible.

The analysis of this surprising phenomenon led to the concept of amenable groups introduced and studied first by von Neumann (1929) [10]. Since that time the subject developed into a field which has importance beside analysis, group theory and geometry in discrete mathematics and computer science (e.g., in the theory of random walks, percolation, expanders).

The Hausdorff-Banach-Tarski paradoxical decompositions of the ball (or of the sphere) in $\mathbb{R}^d$ exist for $d = 3$ (and also for $d > 3$), but do not exist for $d = 1$ and $d = 2$.

Von Neumann discovered that these different phenomena are due to the difference between the isometry groups of $\mathbb{R}^1, \mathbb{R}^2$ and $\mathbb{R}^3$, the latter one is more *"rich"*. He considered a general setting where the basic notions are the finitely additive group invariant measure (or invariant mean) and the paradoxical groups (or amenable groups=non-paradoxical groups).

The objective of the present talk is to give some illustrations and indications of the wide range of topics which developed from the subject mentioned above, providing some motivation of the particular problem considered in the paper of Deuber, Simonovits and Sós [3] and some of its aftermath.

In the paper [3] - - for an arbitrary metric space the concept of *wobbling transformations* (called more recently also *bounded perturbation of the identity*) is introduced.

Definition. Let (X, d) be a metric space, $A, B \subseteq X$. A bijection $f : A \to B$ is called a *wobbling bijection* if

$$\sup_{x \in A} d(x, f(x)) < \infty \ .$$

$A, B \subseteq X$ are called *wobbling equivalent* if there is a wobbling bijection $f : A \to B$.

Definition. The set $A \subseteq X$ is called *wobbling paradoxical* if there is a decomposition

$$A = A_1 \cup A_2, \qquad A_1 \cap A_2 = \emptyset$$

such that A, A_1, A_2 are pairwise wobbling equivalent.

In [3] wobbling paradoxicity is characterized by the following growth condition: For $A \subset X$, $k > 0$ let $N_k(A)$ denote the k-neighbourhood of A:

$$N_k(A) = \{x \in X : d(x, A) \leq k\} \ .$$

Definition. The metric space (X, d) is *doubling*, if there is a $k > 0$ such that

$$|N_k(A)| \geq 2|A| \text{for every finite} A \subset X.$$

Theorem 1 *Let (X, d) be a discrete and countable metric space. (X, d) is wobbling paradoxical if and only if it is doubling.*

In the lecture we surveyed the connection of wobbling paradoxicity to the amenability of groups, to theory of random walks on graphs and groups and some recent applications of the doubling property and wobbling paradoxicity.

A survey paper written jointly with Gábor Elek will appear in a Volume dedicated to the memory of Walter Deuber.

For detailed information and references about the extremely wide area the reader is referred to the excellent books like of Gromov [5], de la Harpe [6], Lubotzky [9], Paterson[11], Wagon [13], Woess [14], and the many survey papers on these subjects, e.g., by Ceccherini-Silberstein, Grigorchuk and de la Harpe [2], Laczkovich [7], [8], Thomassen and Woess [12].

References

[1] Banach, S. and Tarski, A. [1924] Sur la decomposition des ensembles de point en parties respectivement congruents, *Fund. Math.* **6**, 244-277.

[2] Ceccherini-Silberstein, T., Grigorchuk, R. I. and de la Harpe, P. [1999] Amenability and paradoxical decompositions for pseudogroups and discrete metric spaces (Russian), *Tr. Mat. Inst. Steklova* **224**, 68-111.

[3] Deuber, W. A., Simonovits, M. and Sós, V. T. [1995] A note on paradoxical metric spaces, *Studia Sci. Math. Hungar.* **30**, 17-23.

[4] Elek, G. and T. Sós, V., Paradoxical Decompositions and Growth Properties, (2004) submitted

[5] Gromov, M. [1993] Asymptotic invariants of infinite groups, *Volume2 of Geometric Group Theory* Ed. Niblo, G.A., and Roller, M.A. *London Math. Soc. Lecture Notes Series* **182**

[6] de la Harpe, P. [2000] Topics in geometric group theory, *Th University of Chicago Press*

[7] Laczkovich, M., [1994] Paradoxical decompositions: a survey of recent results, in: *First European Congress of Mathematics (Paris, July 6-10, 1992).* Progress in Mathematics, No. 120, Birkhäuser, 1994, Volume II, 159-184.

[8] Laczkovich, Miklós, Paradoxes in measure theory. Handbook of measure theory, Vol. I, II, 83–123, North-Holland, Amsterdam, 2002.

[9] Lubotzky, A. [1994] *Discrete Groups, Expanding Graphs and Invariant Measureas.* Progress in Mathematics, No. 125, Birkhäuser, 1994.

[10] von Neumann, J. [1929] Zur allgemeinen Theorie des Masses, *Fund. Math.* **13**, 73-116.

[11] Paterson, A.L.T. [1988] *Amenability.* Mathematical Surveys and Monographs No. 29, American Mathematical Society, Providence, Rhode Island, 1988.

[12] Thomassen, C. and Woess, W. [1994] Vertex transitive graphs and accessibility, *J. Comb.Theory B* **26**, 1-60.

[13] Wagon, S. [1986] *The Banach-Tarski paradox.* Second edition, Cambridge Univ. Press, 1986. First paperback edition, 1993.

[14] Woess, W., [2001] Random walks on infinite graphs and groups, *Cambridge Tracts in Mathematics*, **138**, Cambridge University Press.

On the Sparse Regularity Lemma

Angelika Steger

(joint work with S. Gerke, Y. Kohayakawa, V. Rödl)

Over the last decades Szemerédi's regularity lemma [17] has proven to be a very powerful tool in modern graph theory. Roughly speaking, the regularity lemma asserts that one can partition a graph G into a constant number of equal-size parts in such a way that most parts are pairwise ε-regular; see [1, 2, 14] for the precise statement of Szemerédi's regularity lemma and some applications. Unfortunately, in its original setting it only gives nontrivial results for dense graphs, that is graphs with $\Theta(n^2)$ edges. In 1996 Kohayakawa [10] and independently Rödl introduced a variant which holds for sparse graphs, provided they satisfy some additional structural conditions (which essentially mean that the graph does not contain too dense spots). However, using this sparse regularity lemma to prove extremal and Ramsey type results similar to the known results in the dense case, requires an additional key step, as Łuczak showed that one cannot directly generalise the methods used for dense graphs to the sparse case, see [12]. The missing step has been formulated as a conjecture by Kohayakawa, Łuczak and Rödl [11], see also [12]. Over the last few years this conjecture has already attracted considerable attention; see [9] and the references therein. One reason for the popularity of the conjecture is its connection with Turán-type problems in random graphs: if the KŁR conjecture is true for a graph H, then asymptotically almost surely (a.a.s.) the number of edges in any H-free subgraph of a binomial random graph $G_{n,p}$ is at most $((1-1/(\chi(H)-1)+\varepsilon)\binom{n}{2}p$ for any $\varepsilon > 0$ as long as $p > C(\varepsilon, H)n^{-1/d_2(H)}$. Here $\chi(H)$ denotes the chromatic number of H, and $d_2(H)$ denotes the 2-density of H. Observe that the bound on the number of edges in an H-free subgraph is essentially best possible since every graph G contains a $(\chi(H)-1)$-partite subgraph with $(1-1/(\chi(H)-1))|E(G)|$ edges. Also the result is not true for much smaller p since then a.a.s. the number of copies of H in $G_{n.p}$ is much smaller than the number of edges; see [9].

This Turán-type result has been established in a series of papers for various special cases, each requiring its own a tailor-made proof. It is now known when $H = K_3$ is a triangle [3], H is a cycle of arbitrary length [4, 7, 8], and when $H = K_4$ is the complete graph on four vertices [11]. If one only considers denser random graphs, where p is about the square root of the conjectured value, then the result is also known to be true for all complete graphs [13, 16].

In fact in their paper [11] Kohayakawa, Łuczak and Rödl not only proved the Turán problem for $H = K_4$, but also outlined a proof strategy based on the sparse regularity lemma which would prove the Turán result for general graphs H, if one could prove an equivalent of the well known embedding lemma for dense graphs in the sparse context as well. They formulated this requirement as a conjecture — the above mentioned KŁR-conjecture.

In the remainder of this abstract we first state the KŁR-conjecture precisely and then report on recent achievements.

Definition 1 *A bipartite graph $B = (U \dot\cup W, E)$ is called (ε, p)-regular if for all $U' \subseteq U$ and $W' \subseteq W$ with $|U'| \geq \varepsilon n$ and $|W'| \geq \varepsilon n$,*

$$\left| \frac{|E(U', W')|}{p \cdot |U'| \cdot |W'|} - \frac{|E(U, W)|}{p \cdot |U| \cdot |W|} \right| \leq \varepsilon.$$

If instead all such $U' \subseteq U$ $W' \subseteq W$ *just satisfy*

$$|E(U', W')| \;\geq\; p \cdot \lambda \cdot |U'| \cdot |W'|$$

for some constant $\lambda > 0$ *the graph* $B = (U \dot\cup W, E)$ *is called* $(\varepsilon, p, \lambda)$*-lower regular.*

Definition 2 *Let* H *be a graph on* l *vertices. An* l*-partite graph* $G = (V_1 \cup \ldots \cup V_l, E)$ *on* l *pairwise disjoint vertex sets* V_i *of size* n *each is called* (H, n, m, ε)*-regular if the graph induced by* V_i, V_j *is* $(\varepsilon, m/n^2)$*-regular whenever* $\{i, j\} \in E(H)$ *and there are no edges between* V_i *and* V_j *otherwise. The set of all* (H, n, m, ε)*-regular graphs is denoted by* $\mathcal{S}(H; n, m, \varepsilon)$*, and* $\mathcal{F}(H; n, m, \varepsilon)$ *is the set of all graphs in* $\mathcal{S}(H; n, m, \varepsilon)$ *not containing* H *as a subgraph.*

The KŁR-conjecture can now be formulated as follows.

Conjecture 3 *Let* H *be an arbitrary graph and* $\beta > 0$*, then there exist positive constants* ε_0*,* C*,* n_0 *such that*

$$|\mathcal{F}(H, n, m, \varepsilon)| < \beta^m \binom{n^2}{m}^{e(H)}$$

for all $m \geq Cn^{2-1/d_2(H)}$*,* $n \geq n_0$ *and* $0 < \varepsilon \leq \varepsilon_0$*, where* $d_2(H) = \max\{\frac{e(F)-1}{v(F)-2} : F \subseteq H,\ v(F) > 2\}$*.*

Note that $\binom{n^2}{m}^{e(H)}$ is the number of graphs which are "blow-ups" of H, and it is not hard to see that it is also asymptotically equal to $|\mathcal{S}(H, n, m, \varepsilon)|$, so the conjecture asserts that only an exponentially small fraction β^m of such graphs are H-free. It was shown by Łuczak that $|\mathcal{F}(H, n, m; \varepsilon)| > 0$ for some graphs H, see [12] where Łuczak is quoted.

The conjecture is easily seen to be true for trees. It is also known to be true for cycles [15] and for the complete graphs $H = K_4$ and K_5 on four respectively five vertices [5, 6].

One of the key difficulties in the proof of the KŁR-conjecture is the fact that for $m = o(n^2)$ the size of a neighbourhood of a vertex is on average $o(n)$. The definition of regularity, however, only deals with linear sized subsets and thus regularity seem to be not inherited by subgraphs induced on the neighborhoods of some vertices. Recently we were able to prove that nevertheless in the sparse case a hereditary version holds as well.

Theorem 4 *For all* $\beta, \varepsilon', \lambda > 0$ *there exist* $\varepsilon, C > 0$ *such that for all* $(\varepsilon, p, \lambda)$*-regular graphs* $B = (U \dot\cup W, E)$ *the following holds. For all* $q \geq C(\lambda p)^{-1}$ *there exist at most* $\beta^q \binom{|U|}{q}$ *sets* $Q \subseteq U$ *such that* (Q, W) *is not* $(\varepsilon', p, \lambda\varepsilon'/32)$*-lower regular.*

This lemma readily implies much shorter and elegant proofs of the results known so far. It can also be used to prove the Turán result for $H = K_6$ and, hopefully, more general results in the near future.

References

[1] B. Bollobás. Modern Graph Theory, *Springer* 1998.

[2] R. Diestel. Graph Theory, *Springer* 1997

[3] P. Frankl and V. Rödl. Large triangle-free subgraphs in graphs without K_4. *Graphs and Combinatorics* 2, 1986, pp. 135–144.

[4] Z. Füredi. Random Ramsey graphs for the four-cycle. *Discrete Mathematics* 126, 1994, pp. 407–410.

[5] S. Gerke, H. J. Prömel, T. Schickinger, A. Steger, and A. Taraz. K_4-free subgraphs of random graphs revisited. Submitted for publication, 2002.

[6] S. Gerke, T. Schickinger, and A. Steger. K_5-free subgraphs of random graphs. *Random Structures & Algorithms*, accepted for publication.

[7] P.E. Haxell, Y. Kohayakawa, and T. Łuczak. Turán's extremal problem in random graphs: forbidding even cycles. *Journal of Combinatorial Theory* B 64, 1995, pp. 273–287.

[8] P. E. Haxell, Y. Kohayakawa, and T. Łuczak. Turán's extremal problem in random graphs: forbidding odd cycles.*Combinatorica* 16(1), 1996, pp. 107–122.

[9] S. Janson, T. Łuczak, and A. Rucinski. Random Graphs. *John Wiley & Sons,* 2000.

[10] Y. Kohayakawa. Szemerédi's regularity lemma for sparse graphs. In: Foundations of Computational Mathematics (Berlin, Heidelberg) (F. Cucker and M. Shub, eds.), *Springer-Verlag,* 1997, pp. 216-230.

[11] Y. Kohayakawa, T. Łuczak, and V. Rödl. On K^4-free subgraphs of random graphs. *Combinatorica* 17(2), 1997, pp. 173–213.

[12] Y. Kohayakawa, and V. Rödl. Regular pairs in sparse random graphs. *Random Structures & Algorithms* 22 (4), 2003, pp. 359-434.

[13] Y. Kohayakawa, V. Rödl, and M. Schacht. The Turán Theorem for random graphs. *Combinatorics, Probability and Computing*, accepted for publication.

[14] J. Komlós and M.Simonovits. Szemérediʼs regularity lemma and its application in graph theory (in Combinatorics - Paul Ersős is eighty, D. Miklós, V.T. Sós, and T. Szőnyi, eds.) Bolyai Society of Mathematical Studies Vol.2, 1996, pp. 295–352, Budapest.

[15] B. Kreuter. Probabilistic versions of Ramsey's and Turán's theorems. Ph. D. thesis, Humboldt-Universität zu Berlin, 1997, Germany.

[16] T. Szabó and V.H. Vu. Turán's Theorem in sparse random graphs. *Random Structures & Algorithms*, accepted for publication.

[17] E. Szemérdi. Regular partitions of graphs. In Problémes Combinatoires et Théorie des Graphes, Colloques Internationaux CNRS Vol.260, 1978, pp. 399–452.

Solving Extremal Problems Using Stability Theorems

Benjamin Sudakov

(joint work with P. Keevash and in part with N. Alon and J. Balog)

In this talk we discuss a 'stability approach' for solving extremal problems. Roughly speaking, it can be described as follows. In order to show that given configuration is a unique optimum for an extremal problem, we first prove an approximate structure theorem for all constructions whose value is close to the optimum and then use this theorem to show that any imperfection in the structure must lead to a suboptimal configuration. To illustrate this strategy, we use the following results.

- Let $T_k(n)$ be the *Turán graph*, i.e., the complete k partite graph on n vertices with class sizes as equal as possible and denote by $t_k(n)$ the number of edges in $T_k(n)$. Then for $k \geq 2$ and sufficiently large n every graph G on n vertices has at most $2^{t_k(n)}$ distinct 2-edge colorings without a monochromatic clique of size $k+1$. Moreover the equality is only possible if $G = T_k(n)$. This settles a conjecture of Yuster. Our proof is based on Szemerédi's regularity lemma together with some additional tools in Extremal Graph Theory, and provide one of the rare examples of a precise result proved by applying this lemma.
- The Fano plane is a 3-uniform hypergraph with 7 triples on 7 vertices whose edges correspond to the lines of the projective plane over the field with two elements. We show that the maximum number of triples on n vertices not

containing a copy of the Fano plane can be obtain by partitioning vertices into two equal parts and taking all the triples which intersect both of them. This confirms a conjecture of V. Sós from 1976 which was also independently proved by Füredi and Simonovits.

- Let $\mathcal{C}_r^{(2k)}$ be the $2k$-uniform hypergraph obtained by letting $P_1, \cdots, P_r$ be pairwise disjoint sets of size k and taking as edges all sets $P_i \cup P_j$ with $i \neq j$. This can be thought of as the 'k-expansion' of the complete graph K_r: each vertex has been replaced with a set of size k. An example of a hypergraph with vertex set V that does not contain $\mathcal{C}_3^{(2k)}$ can be obtained by partitioning $V = V_1 \cup V_2$ and taking as edges all sets of size $2k$ that intersect each of V_1 and V_2 in an odd number of elements. Let $\mathcal{B}_n^{(2k)}$ denote a hypergraph on n vertices obtained by this construction that has as many edges as possible. We prove a conjecture of Frankl, which states that any hypergraph on n vertices that contains no $\mathcal{C}_3^{(2k)}$ has at most as many edges as $\mathcal{B}_n^{(2k)}$.

  Sidorenko has given an upper bound of $\frac{r-2}{r-1}$ for the Turán density of $\mathcal{C}_r^{(2k)}$ for any r, and a construction establishing a matching lower bound when r is of the form $2^p + 1$. We also show that when $r = 2^p + 1$, any $\mathcal{C}_r^{(4)}$-free hypergraph of density $\frac{r-2}{r-1} - o(1)$ looks approximately like Sidorenko's construction. On the other hand, when r is not of this form, we show that corresponding constructions do not exist and improve the upper bound on the Turán density of $\mathcal{C}_r^{(4)}$ to $\frac{r-2}{r-1} - c(r)$, where $c(r)$ is a constant depending only on r.

  To prove these results we use the tools from extremal graph theory, linear algebra, the Kruskal-Katona theorem and properties of Krawtchouck polynomials.

All these results were obtained jointly with P. Keevash and the first one was also obtained jointly with N. Alon and J. Balogh.

Canonical Colourings with Many Colours

Anusch Taraz

(joint work with B. Bollobás, Y. Kohayakawa, V. Rödl, M. Schacht)

Canonical colouring theorems state that, roughly spoken, every colouring of a sufficiently large object exhibits a local pattern of a given size that is coloured in a very regular way. From this point of view, partition theorems such as Ramsey's or van der Waerden's theorem deal with the special case of colourings with a bounded numbers of colours and assert that here, the local pattern can be guaranteed to be monochromatic. The topic of this talk, on the contrary, is to determine conditions that ensure local spots which are *rich in colours*. Our objects of interest will be both cliques in hypergraphs as well as arithmetic progressions on the integers.

Let us begin with arithmetic progressions. The classical theorem of van der Waerden states that every colouring of the first n natural numbers with at most t colours must contain a *monochromatic* k-term arithmetic progression, provided that n is sufficiently large compared to t and k. If no restriction on the number of colours is given, then the canonical colouring theorem by Erdős and Graham [2] states that we must find a *monochromatic* k-AP or an *injective* k-AP; i.e. one which uses pairwise distinct colours.

What condition could guarantee the latter of the two outcomes? It is not enough to merely ask for the colouring to use many colours globally, as can be seen by the following simple example. If $\ell = 3^i \cdot r$, where r isn't divisible by 3, then colour the number ℓ with colour i. Obviously this colouring uses an unbounded number of colours, but it is easy to see that not even an injective 3-AP will appear. Thus we need a stronger requirement on the colourfulness.

Theorem 1 *For every $k \in \mathbb{N}$ and for every $\varepsilon > 0$ there exist integers t and n_0 such that for every $n \geq n_0$ every colouring $\gamma : [n] \to \mathbb{N}$ with the property that*

$$\forall T \subseteq [n] \text{ with } |T| \geq (1-\varepsilon)n : \qquad |\gamma(T)| > t$$

must contain an injective k-term arithmetic progression.

The proof of this theorem is in fact quite short, as it can be based on a quantitative version [3] of Szemerédi's famous density theorem for arithmetic progressions. For graphs and hypergraphs such a density result does not hold and therefore the situation becomes more difficult. Here we are considering colourings of $E(K_n^r)$, the edges of the complete r-uniform hypergraph on n vertices. For the sake of a simpler exposition, we only mention the case $r = 3$ here. Given a family of disjoint vertex sets $V_1, \ldots, V_s$, we say that two edges $e, e' \subset V_1 \cup \cdots \cup V_s$ are of the same type if $|e \cap V_j| = |e' \cap V_j|$ for all $j = 1, \ldots, s$. Generalizing the result in [1], the following theorem asserts, roughly spoken, the existence of colourful canonical colourings which may be 1-partite, 2-partite or 3-partite.

Theorem 2 *For every $k \in \mathbb{N}$ and for every $\varepsilon > 0$ there exist integers t and n_0 such that for every $n \geq n_0$ every colouring $\gamma : E(K_n^{(3)}) \to \mathbb{N}$ with the property that*

$$\forall T \subseteq E(K_n^{(3)}) \text{ with } |T| \geq (1-\varepsilon)\binom{n}{3} : \qquad |\gamma(T)| > t$$

must contain one of the following colourful canonical colourings:

- *there exists a set* V_1 *and an index* $i \in \{1,2,3\}$ *such that* $|V_1| = k$ *and so that two edges contained in* V_1 *receive the same colour only if their* i*-th vertices in* V_1 *are identical, or*
- *there exist sets* V_1, V_2 *and indices* $i \in \{1,2,3\}$ *and* $j \in \{1,2\}$ *such that* $|V_1| = |V_2| = k$ *and so that two edges of the same type receive the same colour only if their* i*-th vertices in* V_j *are identical, or*
- *there exist sets* V_1, V_2, V_3 *and indices* $i \in \{1,2,3\}$ *and* $j \in \{1,2,3\}$ *such that* $|V_1| = |V_2| = |V_3| = k$ *and so that two edges of the same type receive the same colour only if their* i*-th vertices in* V_j *are identical.*

As an application of this theorem we consider (ℓ, H)-local colourings. For fixed integer ℓ and hypergraph H, a colouring of $E(K_n^r)$ is said to be (ℓ, H)-local, if every copy of H in K_n^r is coloured with at most ℓ different colours. Obviously, the larger we choose ℓ, the more colours can appear in an (ℓ, H)-local colouring. We address two questions:

- Given H, what is the largest value of ℓ such that the maximum number of colours used by an (ℓ, H)-local colouring is still bounded?
- Given H, what is the largest value of ℓ such that the maximum number of colours used by an (ℓ, H)-local colouring is still *essentially bounded*?

Here the term *essentially bounded* means the following: for every $\varepsilon > 0$, the colouring is such that after the removal of a suitably chosen ε-fraction of the edges, the remaining edges only use a bounded number of colours.

References

[1] B. Bollobás, Y. Kohayakawa, and R. Schelp, *Essentially infinite colourings of graphs,* Journal of the London Mathematical Society, **61** (2000), no. 3, 658–670.

[2] P. Erdős and R. Rado, *Combinatorial theorems on classifications of subsets of a given set,* Proceedings of the London Mathematical Society, **2** (1952), no. 3, 417–439.

[3] P. Frankl, R.L. Graham, and V. Rödl, *Quantitative versions of combinatorial partition theorems,* Jahresber. Deutsch. Math.-Verein. **92** (1990), no. 3, 130–144.

Chromatic Numbers of Triangle-free Graphs and their Complements
Carsten Thomassen

It is easy to see that triangle-free graphs may have large minimum degree. It is also well-known that they may have arbitrarily large chromatic numbers. Can these two phenomena happen simultaneously? Erdős and Simonovits asked in 1973 for which positive real numbers c, there exists a function $f(c)$ such that the following holds: If G is a triangle-free graph with n vertices and minimum degree at least cn, then the chromatic number is at most $f(c)$. (In other words, the chromatic number is independent of the number of vertices of the graph). They proved that $f(c)$ does not exist for $c < 1/3$. I proved a few years ago that $f(c)$ exists for each $c > 1/3$. So only the case $c = 1/3$ remains open. S. Brandt has conjectured that $f(1/3) = 4$.

Hajos' conjecture says that every graph of chromatic number k contains a subdivision of the complete graph on k vertices. The conjecture was disproved by Catlin in 1979 for all k greater than 6. Kühn and Osthus have verified Hajos' conjecture for graphs of girth greater than 100. The conjecture is open for triangle-free graphs. I showed recently that, if a regular triangle-free graph has bipartite edge-index greater than the number of vertices of the graph, then the complement is a counterexample to Hajos' conjecture. Thus, the complements of triangle-free graphs provide a large class of interesting counterexamples, and it is conceivable that some of these might be counterexamples to Hadwiger's conjecture as well. Searching for possible counterexamples, I tried to investigate the bipartite edge-index of triangle-free graphs on a fixed surface, in particular the projective plane. I found no natural graphs with a sufficiently large bipartite edge-index. Instead I found some with a small bipartite edge-index solving two open problems stated in Bollobas' classical monograph "Extremal Graph Theory" from 1978. One of the problems, due to Erdős, involves the smallest possible bipartite edge-index $g(n)$ of a 4-color-critical graph on n vertices. Erdős asked if $g(n)$ tends to infinity as n tends to infinity. I showed that $g(n)$ equals 3 or 4 for infinitely many n.

Dynamic Configuration of Optical Telecommunication Networks
Andreas Tuchscherer

We investigate methods for online call admission and routing and wavelength assignment in optical telecommunication networks. On demand connections are established by lightpaths which are optical channels that operate on one wavelength and can pass several network links without any opto-electronic conversion.

Definition 1 *An* optical network *is a triple* (G, Λ, W)*, where*

- $G = (V, E)$ *is a simple and undirected graph,*
- $\Lambda = \{\lambda_1, \ldots, \lambda_k\}$ *is a set of wavelengths, and*
- $W : E \to 2^\Lambda$ *is a map from* E *to the power set of* Λ*, where* $W(e)$ *is the set of wavelengths generally available on edge* e*.*

A lightpath *in the optical network* (G, Λ, W) *is a pair* (p, λ) *which consists of a path* p *in* G *together with a wavelength* $\lambda \in \Lambda$ *such that* $\lambda \in W(e)$ *for each edge* $e \in E(p)$*.*

The *Wavelength Division Multiplexing* technique allows for using different wavelengths on one edge simultaneously. However, each wavelength on an edge cannot be used by more than one lightpath at the same time.

Definition 2 (Wavelength conflict constraint) *For each pair of simultaneously routed lightpaths* (p_1, λ_1) *and* (p_2, λ_2) *in an optical network* (G, Λ, W)*, we have:*

$$E(p_1) \cap E(p_2) = \emptyset \text{ or } \lambda_1 \neq \lambda_2.$$

A lightpath (p, λ) *is called* free *if it can be realized without violating the wavelength conflict constraint.*

The considered problem can be formulated as follows.

Definition 3 (Dynamic Singleclass Call Admission Problem) *An instance of the* Dynamic Singleclass Call Admission Problem *(*DSCA*) is given by an optical network* (G, Λ, W)*, a time horizon* T*, and a sequence of connection requests* $(\sigma_1, \sigma_2, \ldots)$ *with* $\sigma_j = (u_j, v_j, b_j, t_j, d_j, p_j)$*, where*

$$\begin{aligned} u_j, v_j \in V &\quad \textit{are the end nodes,} \\ b_j \in \mathbb{N} &\quad \textit{is the number of required lightpaths,} \\ t_j \in [0, T] &\quad \textit{is the start time,} \\ d_j \in \mathbb{R}_+ &\quad \textit{is the duration,} \\ p_j \in \mathbb{R}_+ &\quad \textit{is the profit.} \end{aligned}$$

The task is to maximize the total profit gained such that valid answers are given to all connection requests. The answer for each σ_j *must be given without knowledge of calls with later start times and specifies whether the request is accepted or rejected. If* σ_j *is accepted, it contributes* p_j *to the total profit but requires that* b_j *lightpaths connecting* u_j *and* v_j *are realized in* (G, Λ, W) *from* t_j *until* $t_j + d_j$*. In doing so, the wavelength conflict constraint must be satisfied all the time.*

Concerning the evaluation of online algorithms for the problem DSCA by competitive analysis, the following negative result can easily be shown.

Theorem 4 ([Tuc03]) *For the problem* DSCA *with $d_j = \infty$ and $p_j = b_j$ for each request σ_j, the competitive ratio of each deterministic competitive algorithm is km, where k denotes the number of wavelengths and m denotes the number of edges in the optical network.*

In the following, we report on the practical approach. The algorithms below are evaluated by simulation. The *greedy algorithms* have originally been proposed in [MA98]. We distinguish between two variants: partial wavelength search (PWS) and total wavelength search (TWS).

PWS: Let $\lambda_{i_1}, \ldots, \lambda_{i_k}$ be some order on the set of wavelengths. If there is a free $[u, v]$-lightpath, route a shortest one in wavelength λ, where λ is the first wavelength in the order providing any free $[u, v]$-lightpath.

TWS: Let $\lambda_{i_1}, \ldots, \lambda_{i_k}$ be some order on the set of wavelengths. If there is a free $[u, v]$-lightpath, route a shortest one in wavelength λ, where λ is the first wavelength in the order providing a *globally shortest* free $[u, v]$-lightpath.

Sorting the wavelengths in order of decreasing current availability (number of edges where the wavelength can currently be used) turned out to yield the best versions in partial and total wavelength search (see [Tuc03]). We denote the corresponding algorithms by PACK(P) and PACK(T).

The second class of algorithms (*network fitness algorithms*) have been developed at Konrad-Zuse-Zentrum für Informationstechnik Berlin (ZIB) in a joint project with T-Systems Nova GmbH.

FIT: Let $\mathrm{fit} : \mathcal{S} \to \mathbb{R}_+$ be some network fitness function, where $\mathcal{S}$ denotes the set of all possible network states of (G, Λ, W) (a network state corresponds to a configuration of routed lightpaths). If there is a free $[u, v]$-lightpath, route such a lightpath (p, λ) that the resulting state $S + (p, \lambda)$ yields a maximum fitness value.

We consider two network fitness algorithms called *available-lightpaths-reduction* (ALR) and *single-flow-reduction* (SFR). While ALR defines the fitness as the total number of currently free lightpaths, the algorithm SFR defines the fitness as the sum over all pairs of nodes s and t and each wavelength λ of the maximum number of free edge-disjoint $[s, t]$-lightpaths in wavelength λ.

We have investigated by simulation the *blocking probability* (ratio of rejected requests and appeared requests) depending on the traffic load (multiplex factor).

Figure 2 depicts the results for the four presented algorithms in a setting with randomly generated calls. It turns out that the total wavelength search version PACK(T) is superior to the partial wavelength search version PACK(P) and produces solutions with about the same quality as ALR. The network fitness algorithm SFR performs best.

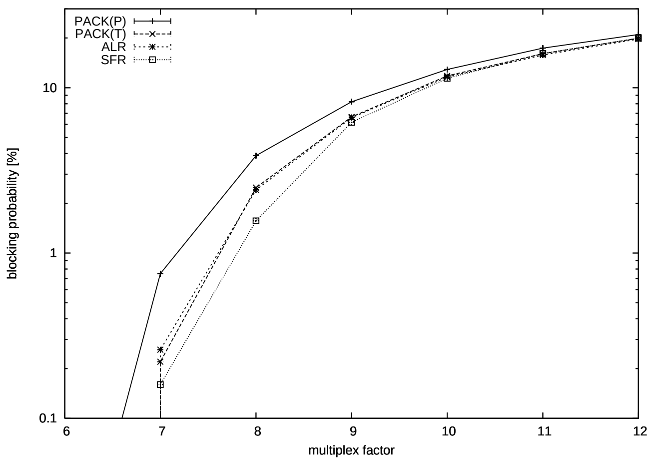

Figure 2: Results of selected algorithms in a 14-nodes network.

References

[MA98] Ahmed Mokhtar and Murat Azizoglu, *Adaptive wavelength routing in all-optical networks*, IEEE/ACM Transactions on Networking **6** (1998), no. 2, 197–206.

[Tuc03] Andreas Tuchscherer, *Dynamical configuration of transparent optical telecommunication networks*, diploma thesis, Technische Universität Berlin, March 2003.

On the Turán Number for the Hexagon

Jacques Verstraëte
(joint work with Zoltan Füredi and Assaf Naor)

One of the fundamental problems in extremal combinatorics is the determination of the maximum number of edges in a graph which contains no $2k$-cycles. The densest constructions of $2k$-cycle-free graphs for certain small values of k arise from the existence of rank two geometries called *generalized k-gons*, first introduced by Tits [5]. These may be defined as rank two geometries whose bipartite incidence graphs are r-regular graphs of diameter k and girth $2k$, where $r > 2$ and $k > 2$, and are known to exist only when k is three, four or six. This fact is an important consequence of a fundamental theorem of Feit and Higman [3]. It is therefore of interest to examine the extremal problem for quadrilaterals, hexagons, and cycles of length ten. In these cases, Lazebnik, Ustimenko and Woldar [4] used the existence of polarities of generalized polygons to construct dense $2k$-cycle-free graphs.

Erdős and Simonovits [2] conjectured the asymptotic optimality of these graphs, by conjecturing that the extremal number for the $2k$-cycle is asymptotic to $\frac{1}{2}n^{1+1/k}$ as n tends to infinity. This was known to hold for quadrilaterals almost fifty years ago, but was recently disproved in [4] for cycles of length ten. The only remaining case allowed by the Feit-Higman theorem is the case of hexagons. In this paper, we refute the Erdős-Simonovits conjecture for hexagons:

Theorem 1 *For infinitely many positive integers n, there are n-vertex hexagon-free graphs of size at least*

$$\frac{3(\sqrt{5}-2)}{(\sqrt{5}-1)^{4/3}}n^{4/3} + O(n) \;\approx\; 0.534n^{4/3}.$$

On the other hand, every n-vertex hexagon-free graph has size at most $\lambda n^{4/3} + O(n)$, where $\lambda \approx 0.627$ is the real root of $16\lambda^3 - 4\lambda^2 + \lambda - 3 = 0$.

The proof of Theorem 1 requires a statement about hexagon-free bipartite graphs, which is interesting in its own right (see de Caen and Szekely [1]).

Theorem 2 *Let m, n be positive integers. Then an m by n bipartite hexagon-free graph has size at most $2^{1/3}(mn)^{2/3} + O(n)$. When $m = 2n$ or $n = 2m$, there are m by n bipartite graphs with $2^{1/3}(mn)^{2/3} + O(n)$ edges.*

References

[1] de Caen, D., Székely, L.A. *The maximum size of* 4- *and* 6-*cycle free bipartite graphs on* m, n *vertices.* Sets, graphs and numbers (Budapest, 1991), 135-142, Colloq. Math. Soc. János Bolyai, 60, Noth-Holland, Amsterdam, 1992.

[2] Erdős, P., Simonovits, M. *Compactness results in extremal graph theory.* Combinatorica 2 (1982), no. 3, 275–288.

[3] Feit, W., Higman, G. The nonexistence of certain generalized polygons. J. Algebra 1 1964 114–131.

[4] Lazebnik, F., Ustimenko, V. A., Woldar, A. J., *Polarities and* $2k$*-cycle-free graphs.* Discrete Math. 197/198 (1999), 503–513.

[5] Tits, J., *Théorème de Bruhat et sous-groupes paraboliques.* C. R. Acad. Sci. Paris 254 (1962), 2910–2912.

Sharp Bounds on Long Arithmetic Progressions in Sumsets

V. H. Vu

(joint work with E. Szemerédi)

One of the main tasks of additive number theory is to examine structural properties of sumsets. For a set A of integers, the sumset $lA = A + \cdots + A$ consists of those numbers which can be represented as a sum of l elements of A. A closely related notion is that of l^*A, which is the collection of numbers which can be represented as a sum of l different elements of A. Among the most well-known results in all mathematics are Vinogradov's theorem which says that $3\mathbb{P}$ ($\mathbb{P}$ is the set of primes) contains all sufficiently large odd number and Waring's conjecture (proved by Hilbert, Hardy and Littlewood, Hua, and many others) which asserts that for any given r, there is a number l such that $l^*\mathbb{N}^r$ ($\mathbb{N}^r$ denotes the set of r^{th} powers) contains all sufficiently large positive integers (see [16] for an excellent exposition concerning these results).

In recent years, a considerable amount of attention has been paid to the study of finite sumsets. For a finite set A, the natural analogue of Vinogadov-Waring results is to show that under proper conditions, a finite set sumset lA (l^*A) contains a long arithmetic progression.

Let us assume that A is a subset of the interval $[n] = \{1, \ldots, n\}$, where n is a large positive integer. The concrete problem we would like to talk about is to estimate the length of the longest arithmetic progression in lA (l^*A) as a function of l, n and $|A|$ (we are, of course, talking about the worst set A). This problem was stated explicitly for the sumset lA in a survey of Freiman, but we notice that many results had been proved earlier [1, 11, 12, 5]. We adapt a notation from Freiman's paper and denote by $f(|A|, l, n)$ the minimum length of the longest arithmetic progression in lA, where the minimum is taken over all sets $A \subset [n]$ with $|A|$ elements ($f^*(|A|, l, n)$ is defined similarly).

In this paper, we solve the problem completely for a wide range of l and $|A|$. In fact, our method carries us far beyond our original aim of estimating $f(|A|, l, n)$ and $f^*(|A|, l, n)$. We are able to show that lA and l^*A not only contain large arithmetic progressions, but also large proper generalized arithmetic progressions. Let us state the result for lA.

Theorem 1 *For any fixed positive integer d there are positive constants C and c depending on d such that the following holds. For any positive integers n and l and any set $A \subset [n]$ satisfying $l^d|A| \geq Cn$, lA contains an arithmetic progression of length $cl|A|^{1/d}$.*

Corollary 2 *For any fixed positive integer d there are positive constants C_1, C_2, c_1 and c_2 depending on d and ϵ such that whenever $\frac{C_1 n}{l^d} \leq |A| \leq \frac{C_2 n}{l^{d-1}}$*

$$c_1 l|A|^{1/d} \leq f(|A|, l, n) \leq c_2 l|A|^{1/d}.$$

Theorem 3 *For any fixed positive integer d there are positive constants C and c depending on d such that the following holds. For any positive integers n and l and any set $A \subset [n]$ satisfying $l^d|A| \geq Cn$, lA contains a proper GAP of rank d' and volume at least $cl^{d'}|A|$, for some $d' \leq d$.*

The same results hold for l^*A. However the proofs are much more difficult because of the assumption that the elements in a sum must be different. We can also prove similar results for finite fields.

Our results have some interesting applications. In particular, we settle two forty year old conjectures of Erdős [3] and Folkman [7] (respectively) concerning infinite arithmetic progressions. Let us end this abstract with the statements of these conjectures/theorems. For an infinite sequence of integers A, S_A denotes the collection of partial sums of A.

Theorem 4 *Let $A = a_1 < a_2 < \dots$ be a sequence of positive integers with density at least $Cn^{1/2}$, where C is a sufficiently large constant. Then S_A contains an infinite arithmetic progression.*

This theorem was conjectured by Folkman in 1966 [7] and was a refined form of an earlier conjecture by Erdős made in 1962 [3] (see also [4] and [10] for more recent discussions).

Theorem 5 *Let $A = a_1 < a_2 < \dots$ be a sequence of positive integers with density at least Cn, where C is a sufficiently large constant. Then S_A contains an infinite arithmetic progression.*

By the density of A, we mean the number of elements of A between 1 and n. In the second theorem, this number may be large than A as we allow repetitions. It is known since the sixties (see [2]) that both statements are sharp, up to the constant C.

Most of the results discussed here appear in [14] and [15]. A related paper is [13], in which an application of different kind is discussed.

References

[1] J. Bourgain, On arithmetic progressions in sums of sets of integers. A tribute to Paul Erds, 105–109, Cambridge Univ. Press, Cambridge, 1990.

[2] J.W.S Cassels, On the representation of integers as the sums of distinct summands taken from a fixed set, *Acta Sci. Math. Szeged* 21 1960 111–124.

[3] P. Erdős, On the representation of large interges as sums of distinct summands taken from a fixed set, *Acta. Arith.* 7 (1962), 345-354.

[4] P. Erdős and R. Graham, **Old and new problems and results in combinatorial number theory**. Monographies de L'Enseignement Mathatique [Monographs of L'Enseignement Mathatique], 28. Universitde Gene, L'Enseignement Mathatique, Geneva, 1980. 128 pp.

[5] G. Freiman, H. Halberstam and I. Ruzsa, Integer sum sets containing long arithmetic progressions, *J. London Math. Soc.* (2) 46 (1992), no. 2, 193–201.

[6] G. Freiman, New analytical results in subset-sum problem. Combinatorics and algorithms (Jerusalem, 1988). *Discrete Math.* 114 (1993), no. 1-3, 205–217.

[7] J. Folkman, On the representation of integers as sums of distinct terms from a fixed sequence, *Canad. J. Math.* 18 1966 643–655.

[8] R. Graham, Complete sequences of polynomial values, *Duke Math. J.* 31 (1964), 275–286.

[9] B. Green, Arithmetic progressions in sumsets, *Geom. Funct. Anal.* 12 (2002), no. 3, 584–597

[10] C. Pomerance and A. Sárközy, Combinatorial number theory, **Handbook of combinatorics**, Vol. 1, 2, 967–1018, Elsevier, Amsterdam, 1995.

[11] I. Ruzsa, Generalized arithmetical progressions and sumsets, *Acta Math. Hungar.* 65 (1994), no. 4, 379–388.

[12] A. Sárközi, Finite addition theorems I, *J. Number Theory,* 32, 1989, 114–130.

[13] E. Szemerédi and V.H. Vu, Long arithmetic progressions in sumbsets and the number of x-sum-free sets, *submitted*. An electronic copy is available at `http://www.math.ucsd.edu/~vanvu/`.

[14] E. Szemerédi and V.H. Vu, Finite and Infinite arithmetic progressions in sumsets, *submitted*. An electronic copy is available at `http://www.math.ucsd.edu/~vanvu/`.

[15] E. Szemerédi and V.H. Vu, Sharp bounds on arithmetic progressions in sumsets, *manuscript*. An electronic copy is available at `http://www.math.ucsd.edu/~vanvu/`.

[16] R. Vaughan, **The Hardy-Littlewood method**. Second edition. Cambridge Tracts in Mathematics, 125. Cambridge University Press, Cambridge, 1997.

On Musin's Proof for the Kissing Number in Dimensions 3 and 4

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The "kissing number problem" asks for the maximal number of white spheres that can touch a black sphere of the same size in n-dimensional space. The answers in dimensions one, two and three are classical, while the answers in dimensions eight and twenty-four were a big surprise in 1979, based on an extremely elegant method initiated by Philippe Delsarte in the early eighties.

However, despite the fact that in dimension four there is a really special configuration which is conjectured optimal—the shortest vectors in the D_4 lattice, which are also the vertices of a regular 24-cell—it was even *proved* [1] that the bounds given by Delsarte's method aren't good enough to solve the problem in dimension four: This may explain the astonishment even to experts when last fall Oleg Musin announced a solution (currently under review) of the problem, based on a clever modification of Delsarte's method [3, 4].

The purpose of my talk was to outline Musin's new ideas. This started with a short description of the classical approach, due to Delsarte, Goethals & Seidel [2]: If $f(t) = \sum_k c_k G_k^{(n)}(t)$ is a non-negative combination of Gegenbauer polynomials which satisfies $f(t) \leq 0$ in the range $t \in [-1, \frac{1}{2}]$, then $\kappa(n) \leq f(1)/c_0$ is an upper bound for the kissing number in dimension n. Musin's modification is to require the condition $f(t) \leq 0$ only in a range $t \in [t_0, \frac{1}{2}]$ for some fixed $t_0 < -\frac{1}{2}$, while $f(t)$ must be strictly monotonically decreasing in the range $t \in [-1, t_0]$. This leads to an upper bound on $\kappa(n)$ in terms of some non-convex non-linear optimization problems. Musin explains ideas that reduce the dimensions of these optimization problems considerably. Apparently the problems are rather well-behaved, and can be solved numerically.

Their solution not only yields $\kappa(4) = 24$, but it also gives us a systematic and conceptual new proof for the Newton–Gregory problem, $\kappa(3) = 12$, which was first resolved by Schütte and van der Waerden (1953).

References

[1] V. V. Arestov and A. G. Babenko, *Estimates for the maximal value of the angular code distance for* 24 *and* 25 *points on the unit sphere in* $\mathbb{R}^4$, Math. Notes, 68 (2000), pp. 419–435.

[2] P. Delsarte, J. M. Goethals, and J. J. Seidel, *Spherical codes and designs*, Geomeetriae Dedicata, 6 (1977), pp. 363–388.

[3] O. R. Musin, *The kissing number in four dimensions.* Preprint, September 2003, 22 pages; `math.MG/0309430`.

[4] ——, *The problem of the twenty-five spheres*, Russian Math. Surveys, 58 (2003), pp. 794–795.

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