MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

Report No. 5/2004

Wave Motion

Organised by
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Joachim Escher (Hannover)

January 25th – January 31th, 2004

Introduction by the Organisers

The workshop **Wave Motion** that took place in the period January 25–31, 2004 was devoted to the study of nonlinear wave phenomena. The modelling of waves leads to a variety of difficult mathematical issues, involving several domains of mathematics: partial differential equations, harmonic analysis, dynamical systems, topological degree theory.

The progam of the workshop consisted in 18 talks, presented by international specialists in nonlinear waves coming from England, France, Germany, Japan, Norway, Sweden, Switzerland, U.S.A., and by three discussion sessions on the topics "Open Problems in PDEs", "Stability Phenomena in the Theory of Nonlinear Waves", and "Geodesic Flows and Fluid Mechanics". Moreover, several doctoral and post-doctoral fellows participated in the workshop and did benefit from the unique academic atmosphere at the Oberwolfach Institute.

The proceedings of the workshop "Wave Motion" will appear as a special issue of the *Journal of Nonlinear Mathematical Physics*.

Workshop on Wave Motion

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Abstracts

Nonlinear water waves and spatial dynamics

MARK D. GROVES

The water-wave problem is the study of the three-dimensional irrotational flow of a perfect fluid bounded below by a rigid horizontal bottom $\{y = 0\}$ and above by a free surface $\{y = h + \eta(x, z, t)\}$ subject to the forces of gravity and surface tension. This remarkable problem, first formulated in terms of a potential function ϕ by Euler (Figure 1), has become a paradigm for most modern methods in nonlinear functional analysis and nonlinear dispersive wave theory. Its mathematical study has historically called upon many different approaches (iteration methods, bifurcation theory, complex variable methods, PDE methods, the calculus of variations, positive operator theory, topological degree theory, KAM theory, symplectic geometry, . . .). In this talk I would like to illustrate the role of the water-wave problem as a paradigm in the theory of Hamiltonian systems and conservative pattern-formation problems.

$$
\begin{aligned}
&\phi_{zz} + \phi_{yy} + \phi_{zz} = 0, && 0 < y < h + \eta,\\
&\phi_y = 0, && y = 0,\\
&\phi_y = \eta_t + \eta_x\phi_x + \eta_z\phi_z, && y = h + \eta,\\
&\phi_t = -\frac{1}{2}(\phi_x^2 + \phi_y^2 + \phi_z^2) - g\eta &&\\
&\qquad + \sigma\left[\frac{\eta_x}{\sqrt{1 + \eta_x^2 + \eta_z^2}}\right]_x + \sigma\left[\frac{\eta_z}{\sqrt{1 + \eta_x^2 + \eta_z^2}}\right]_z, && y = h + \eta
\end{aligned}
$$

FIGURE 1. Euler (1707–1783), who first formulated the water-wave problem (left)

Travelling water waves are solutions of the water-wave problem which are stationary in a uniformly translating reference frame, so that $\eta(x, z, t) = \eta(\xi, z)$, where $\xi = x - ct$. The resulting time-independent problem can be approached using the method of *spatial dynamics*, which was devised by K. Kirchgässner specifically with water waves in mind and has now found applications in a huge range of other problems (reaction-diffusion equations, spiral waves, mathematical biology, . . .). The idea is to formulate a stationary problem as an evolutionary equation in which an unbounded *spatial* coordinate plays the role of the time-like variable. In the travelling water-wave problem one can take any horizontal direction $X = \sin\theta_2\,\xi - \cos\theta_2\,z$ as the time-like variable and formulate the equations as an evolutionary equation

$$u_X = Lu + Nu, \qquad u \in \mathcal{X}; \tag{1}$$

the infinite-dimensional phase space $\mathcal{X}$ is constructed to contain functions which are, for example, $2\pi/\nu$-periodic in a second, different horizontal direction $Z = \sin\theta_1\,\xi - \cos\theta_1\,z$.

The evolutionary equation (1) is found by performing a Legendre transform upon the classical variational principle

$$\delta \int \int_0^{2\pi} \left\{ \int_0^{h+\eta} \Big(-\sin\theta_2\,\phi_X - \nu\sin\theta_1\,\phi_Z + \frac{1}{2}\left(\phi_X^2 + \phi_y^2 + \nu^2\phi_Z^2 + 2\nu\cos(\theta_1-\theta_2)\phi_X\phi_Z\right)\Big)\,dy \right.$$
$$\left. + \frac{1}{2}g\eta^2 + \sigma\left(\sqrt{1+\eta_X^2+\nu^2\eta_Z^2+2\nu\cos(\theta_1-\theta_2)\eta_X\eta_Z} - 1\right)\right\}\,dZ\,dX = 0$$

for the desired wave motions. In many cases equation (1) can be treated using an invariant-manifold theory due to A. Mielke, which was again developed with this problem in mind, but is now used in a wide variety of problems (elasticity, solid mechanics, ...). This theory shows that all small, bounded solutions lie on a finite-dimensional invariant manifold and thus reduces the water-wave problem to a locally equivalent finite-dimensional Hamiltonian system; the dimension and character of this reduced system depend upon the values of the physical parameters (gravity g, surface tension σ, wave speed c, water depth h).

Two-dimensional i.e. z-independent travelling waves lend themselves naturally to an application of the spatial dynamics method with $X = \xi$. B. Buffoni, M. D. Groves & J. F. Toland showed that in a certain parameter regime the invariant manifold is four dimensional and controlled by the Hamiltonian equation

$$u'''' + Pu'' + u - u^2 = 0, \qquad P \in (-2, -2+\epsilon).$$

Amazingly, this equation turns up in many, seemingly unrelated problems in applied science, for example in nonlinear elasticity, nonlinear optics and now nonlinear water waves. One of its most interesting features is that it exhibits *chaotic behaviour:* there is a Smale-horseshoe structure in its solution set. As a consequence, it has infinitely many *homoclinic solutions*, that is solutions which decay to zero as the time-like variable tends to infinity. The corresponding solutions of the water-wave problem are called *solitary waves* and decay to the undisturbed state of the water as $\xi \to \pm\infty$. This result shows that there are infinitely many of them; they are waves of depression with $2, 3, 4, \ldots$ large troughs separated by $2, 3, \ldots$ small oscillations, and their oscillatory tails decay exponentially to zero. Two waves from this family are sketched in Figure 2.

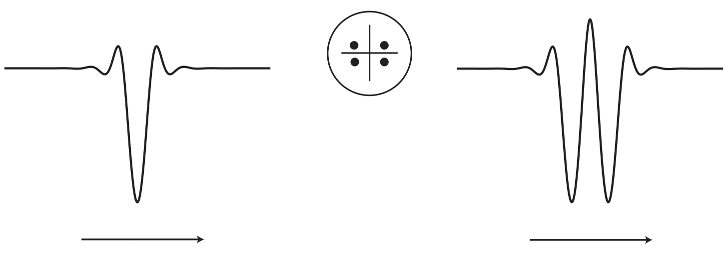


FIGURE 2. Two of the multi-troughed solitary waves found by B. Buffoni, M. D. Groves & J. F. Toland on a four-dimensional invariant manifold with the depicted eigenvalue structure.

The study of two-dimensional solitary waves was continued by G. Iooss & K. Kirchgässner and B. Buffoni & M. D. Groves, who noticed that there are parameter values for which the invariant manifold is four-dimensional and a *Hamiltonian-Hopf bifurcation* takes place (two nonsemisimple imaginary eigenvalues become complex as a parameter is varied). Hamiltonian-Hopf bifurcations are well-known to researchers in the field of celestial mechanics, where they occur in the restricted three-body problem for the planar motion of a light body orbiting the centre of mess of two heavy bodies; the Hamiltonian-Hopf bifurcation occurs for a certain value of the mass ratio of the two heavy particles (Routh's ratio). Iooss & Kirchgässner used the Birkhoff normal form to show that Hamiltonian-Hopf bifurcations generate homoclinic solutions which take the form of periodic wave trains modulated by exponentially decaying envelopes (Figure 3). Buffoni & Groves showed that there are in fact infinitely many such solutions which resemble multiple copies of Iooss & Kirchgässner's solutions; their proof is based upon modern methods from the calculus of variations (mountain-pass arguments and the concentration-compactness principle) and the topological degree. These results are not restricted to the water-wave problem in which they emerge; they provide dramatic new solutions to the three-body problem and indeed Hamiltonian-Hopf bifurcations have been detected in a range of situations (Taylor-Couette flows, nonlinear elasticity,...).

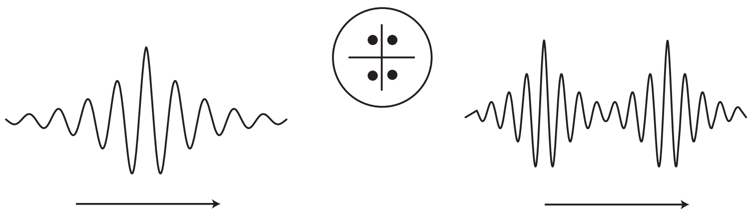


FIGURE 3. Two of the multi-packet solitary waves found by B. Buffoni & M. D. Groves on a four-dimensional invariant manifold with the depicted eigenvalue structure.

M. D. Groves & M. Haragus have recently classified all the possible bifurcation scenarios for *three-dimensional* travelling waves using the spatial dynamics method. In particular, they compiled a catalogue of three-dimensional waves which have solitary-wave or *generalised solitary-wave* profiles in a distinguished horizontal direction (the time-like direction); these profiles decay respectively to zero and to a periodic wavetrain at large distances. Some of the waves are rather exotic, as Figure 4 shows.

Groves & Haragus also examined doubly periodic travelling waves using spatial dynamics. Periodicity in the Z-direction is built into the method, so that doubly periodic waves are found as solutions of the reduced Hamiltonian system which are periodic in the time-like direction X. Such solutions are found using the classical Lyapunov centre theorem, and depending upon the physical parameters one encounters all possible cases: nonresonant eigenvalues, semisimple eigenvalue resonances, nonsemisimple eigenvalue resonances and equal or opposite Krein signatures! Doubly periodic surface waves (Figure 5) and periodic motion of heavenly bodies (the n-body problem in celestial mechanics) are, according to the above observations, two aspects of the same mathematical theory, namely finite-dimensional Hamiltonian systems and the Lyapunov centre theorem.

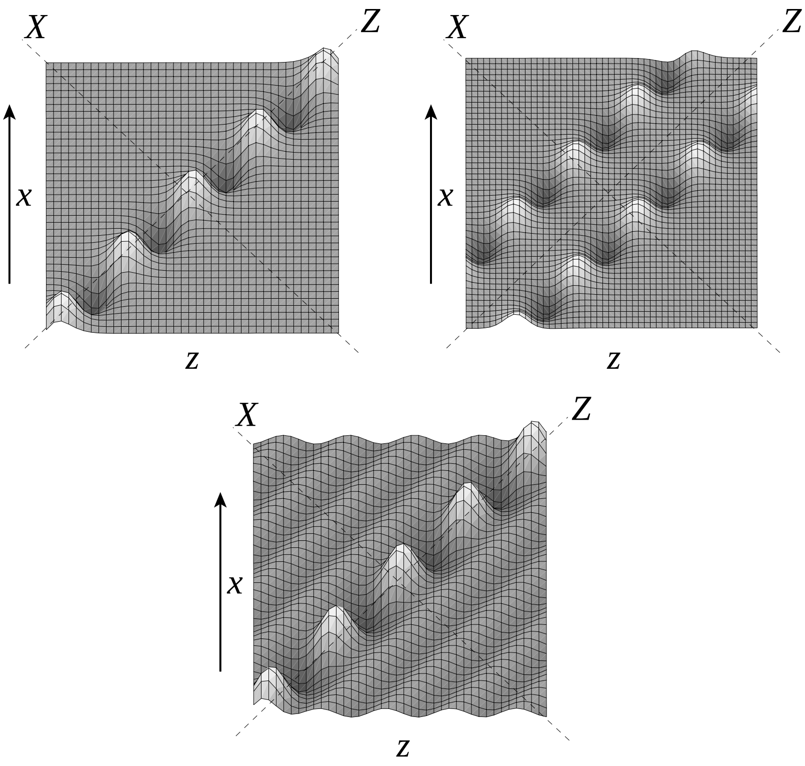
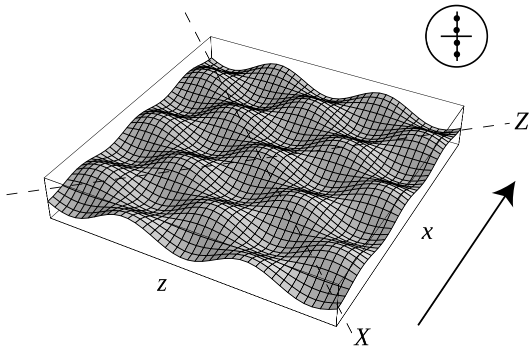


FIGURE 4. Three examples from the catalogue of three-dimensional travelling waves compiled by M. D. Groves & M. Haragus. These waves have the profile of (a) a one-pulse solitary wave, (b) a two-pulse solitary wave and (c) a generalised solitary-wave in one distinguished spatial direction (X) and are periodic in another (Z); they move with constant speed and without change of shape in the x direction (arrowed).



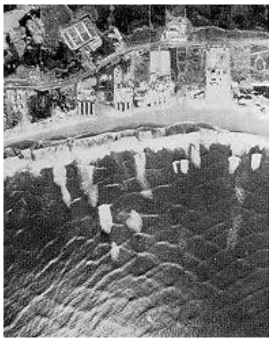


FIGURE 5. The doubly periodic wave on the left is constructed using the Lyapunov centre theorem on a four-dimensional invariant manifold. Hexagonal doubly periodic waves are often seen in nature, as this ariel photograph on the right shows; they can be explained mathematically by this procedure.

Corner defects in almost planar interface propagation

MARIANA HARAGUS
(joint work with Arnd Scheel)

We study existence and stability of almost planar interfaces in reaction-diffusion systems. Almost planar here refers to the angle of the interface at each point, relative to a fixed planar interface. Most of the interfaces that we construct are planar at infinity, with possibly different orientations at $+\infty$ and $-\infty$ in an arclength parameterization. We refer to all these types of interfaces as *corner defects*. According to their angles at $\pm\infty$ we distinguish between interior corners, exterior corners, steps and holes.

We construct corner defects as perturbations of a planar interface. Assumptions are solely on the existence of a primary planar travelling-wave solution and spectral properties of the linearization at the planar wave. All interfaces that we construct are stationary or time-periodic patterns in an appropriately comoving frame. The corner typically but not always points in the direction opposite to the direction of propagation. In addition, we give stability results which show that "open" classes of initial conditions actually converge to the corner-shaped interfaces we constructed before. The results are stated for reaction-diffusion systems but the method is sufficiently general to cover different situations, as well. In particular we do not rely on monotonicity arguments or comparison principles such that we can naturally include the case of interfaces separating patterned states from spatially homogeneous states.

The method we use is based on the (essentially one-dimensional) dynamical systems approach to the existence of bounded solutions to elliptic equations in cylinders introduced by Kirchgässner. The main idea is to consider an elliptic equation, posed on the (x, y)-plane in a neighborhood of an x-independent wave $q_*(y)$ as a dynamical system in the x-variable and rely on dynamical systems tools such as center-manifold reduction and bifurcation theory to construct bounded solutions to the elliptic equation in a neighborhood of the original wave. Nontrivial, that is non-equilibrium, x-"dynamics" then correspond to nontrivial x-profiles. In the present work, we extend these ideas, incorporating the shift of the y-profile $q_*(y)$ into the reduced dynamics. We then respect this affine action of the symmetry group in the construction and parameterization of the center-manifold such that the reduced equations take a skew-product form. The reduced ordinary differential equation can be viewed as the travelling-wave equation to a viscous conservation law or variants of the Kuramoto-Sivashinsky equation.

Invariance principle and the inverse problems for the periodic Camassa-Holm equation

EVGENI KOROTYAEV

Consider the nonlinear mapping $F : H_1 \to H$ given by

$$F(y) = y' + u_1(y) + u_2(Jy) - u_0(y, y'), \quad Jy = \int_0^x y(s)ds, \quad y \in H_1$$

where the Hilbert spaces $H = \{q : q \in L^2_R(\mathbb{T}),\ \int_0^1 q(x)dx = 0\}$ and $H_1 = \{y, y' \in H\}$. Here the functions u_1, u_2 are real analytic and $u_2'(y) \leq 0$: the constant u_0 is such that $F(y) \in H$. We prove that the map F is a real analytic isomorphism. Furthermore, a priori two-sided estimates of norms of $F(y), y$ are obtained. We apply these results to the inverse problems for the Schrödinger operators $S = -d^2dx^2 + F(y)$ in $L^2(\mathbb{R})$ with a 1-periodic potential $F(y)$. The inverse problem for the operator $S_0S = -d^2dx^2 + p$, (p is periodic) is the well known fact. Thus we have the factorization, which yields the solution the inverse problem for S, with "variable" y. We call these result invariance principle since the results about the inverse problems depend on only the large conditions on the functions u_1, u_2.

In the second part we use this result to study the Camassa-Holm equation. We consider the periodic weighted operator $Ty = -\rho^{-2}(\rho^2 y')' + 14\rho^{-4}$ in $L^2(\mathbb{R}, \rho^2 dx)$ where ρ is a 1-periodic positive function satisfying $q = \rho'/\rho \in L^2(0, 1)$. The spectrum of T consists of intervals separated by gaps. Using the Liouville transform we get the Schrödinger operators $-d^2dx^2 + F$, where $F = q' + q^2 + u_2(Jq) - u_0$, i.e.,in this case $u_1(q) = q^2,\ \ u_2(Jq) = 14e^{-4Jq}$. Firstly, we construct the Marchenko-Ostrovski mapping $q \to$

$h(q)$ and solve the corresponding inverse problem. For our approach it is essential that the mapping h has the factorization $h(q) = h^0(F(q))$, where $q \to F(q)$ is a certain nonlinear mapping and $V \to h^0(F)$ is the Marchenko-Ostrovski mapping for the Hill operator. Secondly, we solve the inverse problem for the gap length mapping and we obtain the trace formula for T.

Noncommutative Deformation of Solitons

OLAF LECHTENFELD

A noncommutative (Moyal) deformation of a function space over $\mathbb{R}^{2n}$ is achieved formally by subjecting the coordinate functions $(x^\mu)_{\mu=1,\dots,2n}$ to the (Heisenberg) algebra

$$x^\mu \star x^\nu - x^\nu \star x^\mu = \mathrm{i}\theta^{\mu\nu},$$

where '$\star$' denotes the deformed product and $(\theta^{\mu\nu})$ is a constant antisymmetric matrix. This induces an associative product on $C^\infty(\mathbb{R}^{2n})$ involving arbitrary powers of a bidifferential operator. A convenient realization of the deformed function algebra trades the coordinate dependence (and the deformed product) for operator valuedness (and the standard compositional product), the operators acting on an auxiliary Fock space such as $L^2(\mathbb{R}^n)$. The model is quantum mechanics.

The generalization of integrable differential equations, for instance the sine-Gordon equation, to the noncommutative setup is ambiguous, but a distinguished choice arises from demanding the existence of a noncommutative Lax pair. The rewriting of the nonlinear differential equation as the compatibility condition of a linear system does not notice the noncommutative deformation since one is already dealing with matrix-valued, i.e. noncommuting, objects. Likewise, established methods for generating solutions to the linear system, e.g. the dressing method, can be deformed painlessly.

We demonstrate this stategy for the example of the U(m) principal chiral (or nonlinear sigma-)

model in two dimensions which allows for solitonic solutions to its equation of motion. The dressing method reduces the task to solving an eigenvalue problem for a linear differential operator which in the noncommutative situation is realized by a simple linear operator in $L^2(\mathbb{R})$. After picking a basis in this Fock space, the noncommutative deformation formally amounts to replacing $n\times n$ matrices by semi-infinite ones. It is therefore not surprising to find not only smooth deformations of the commutative solitonic solutions but also a class of new solitons (even for the U(1) case!) which display a singular commutative ($\theta \to 0$) limit.

Like in the commutative case, all known integrable equations in 3, 2, and 1 dimensions descend from the four-dimensional self-dual Yang-Mills equations by dimensional reduction and specialization. The noncommutative deformation, however, is not compatible with picking any subgroup of U(m). Nevertheless, the method presented here can be applied to any integrable system, such as NLS, KdV, KP, Burgers, Boussinesq, sine-Gordon, or Camassa-Holm. The resulting noncommutative equations will be nonlocal (featuring an infinite number of derivatives) but of a controlled kind; their solutions will have many properties in common with the standard solitons.

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Renormalized entropy solutions for quasilinear anisotropic degenerate parabolic equations

Kenneth Hvistendahl Karlsen
(joint work with Mostafa Bendahmane)

We consider the Cauchy problem for quasilinear anisotropic degenerate parabolic equations with L^1 data. This convection–diffusion type problem is of the form

$$\partial_t u + \mathrm{div} f(u) = \nabla \cdot (a(u)\nabla u) + F, \qquad u(0,x) = u_0(x), \tag{1}$$

where $(t,x) \in (0,T) \times \mathbf{R}^d$; $T > 0$ is fixed; div and ∇ are with respect to $x \in \mathbf{R}^d$; and $u = u(t,x)$ is the scalar unknown function that is sought. The (initial and source) data $u_0(x)$ and $F(t,x)$ satisfy

$$u_0 \in L^1(\mathbf{R}^d), \qquad F \in L^1((0,T) \times \mathbf{R}^d). \tag{2}$$

The diffusion function $a(u) = (a_{ij}(u))$ is a symmetric $d \times d$ matrix of the form

$$a(u) = \sigma(u)\sigma(u)^\top \geq 0, \qquad \sigma \in \left(L^\infty_{\mathrm{loc}}(\mathbf{R})\right)^{d\times K}, \qquad 1 \leq K \leq d, \tag{3}$$

and hence has entries

$$a_{ij}(u) = \sum_{k=1}^{K} \sigma_{ik}(u)\sigma_{jk}(u), \qquad i,j = 1,\ldots,d.$$

The inequality in (3) means that for all $u \in \mathbf{R}$

$$\sum_{i,j=1}^{d} a_{ij}(u)\lambda_i\lambda_j \geq 0, \qquad \forall \lambda = (\lambda_1,\ldots,\lambda_d) \in \mathbf{R}^d.$$

Finally, the convection flux $f(u)$ is a vector–valued function that satisfies

$$f(u) = (f_1(u),\ldots,f_d(u)) \in \left(\mathrm{Lip}_{\mathrm{loc}}(\mathbf{R})\right)^d. \tag{4}$$

It is well known that (1) possesses discontinuous solutions and that weak solutions are not uniquely determined by their initial data (the scalar conservation law is a special case of (1)). Hence (1) must be interpreted in the sense of entropy solutions [15, 19, 20]. In recent years the isotropic diffusion case, for example the equation

$$\text{(5)} \qquad \partial_t u + \operatorname{div} f(u) = \Delta A(u), \quad A(u) = \int_0^u a(\xi)\, d\xi, \quad 0 \le a \in L^\infty_{\text{loc}}(\mathbf{R}),$$

has received much attention, at least when the data are regular enough (say $L^1 \cap L^\infty$) to ensure $\nabla A(u) \in L^2$. Various existence results for entropy solutions of (5) (and (1)) can be derived from the work by Vol'pert and Hudjaev [20]. Some general uniqueness results for entropy solutions have been proved in the one-dimensional context by Wu and Yin [21] and Bénilan and Touré [2]. In the multi-dimensional context a general uniqueness result is more recent and was proved by Carrillo [6, 5] using Kružkov's doubling of variables device. Various extensions of his result can be found in [4, 12, 13, 14, 16, 17, 18], see also [7] for a different approach. Explicit "continuous dependence on the nonlinearities" estimates were proved in [10]. In the literature just cited it is essential that the solutions u possess the regularity $\nabla A(u) \in L^2$. This excludes the possibility of imposing general L^1 data, since it is well known that in this case one cannot expect that much integrability.

The general anisotropic diffusion case (1) is more delicate and was successfully solved only recently by Chen and Perthame [9]. Chen and Perthame introduced the notion of kinetic solutions and provided a well posedness theory for (1) with L^1 data. Using their kinetic framework, explicit continuous dependence and error estimates for $L^1 \cap L^\infty$ entropy solutions were obtained in [8]. With the only assumption that the data belong to L^1, we cannot expect a solution of (1) to be more than L^1. Hence it is in general impossible to make distributional sense to (1) (or its entropy formulation). In addition, as already mentioned above, we cannot expect $\sqrt{a(u)}\nabla u$ to be square-integrable, which seems to be an essential condition for uniqueness. Both these problems were elegantly dealt with in [9] using the kinetic approach.

The purpose of the present paper is to offer an alternative "pure" L^1 well posedness theory for (1) based on a notion of renormalized entropy solutions and the classical Kružkov method [15]. The notion of renormalized solutions was introduced by DiPerna and Lions in the context of Boltzmann equations [11]. This notion (and a similar one called entropy solutions) was then adapted to nonlinear elliptic and parabolic equations with L^1 (or measure) data by various authors. We refer to [3] for some recent results in this context and a list of relevant references. Bénilan, Carrillo, and Wittbold [1] introduced a notion of renormalized Kružkov entropy solutions for scalar conservation laws with L^1 data and proved the existence and uniqueness of such solutions. Their theory generalizes the Kružkov well posedness theory for L^∞ entropy solutions [15].

Motivated by [1, 3] and [9], we introduce herein a notion of renormalized entropy solutions for (1) and prove its well posedness. Let us illustrate our notion of an L^1 solution on the isotropic diffusion equation (5) with initial data $u|_{t=0} = u_0 \in L^1$. To this end, let $T_l : \mathbf{R} \to \mathbf{R}$ denote the truncation function at height $l > 0$ and let $\zeta(z) = \int_0^z \sqrt{a(\xi)}\, d\xi$. A renormalized entropy solution of (5) is a function $u \in L^\infty(0,T; L^1(\mathbf{R}^d))$ such that (i) $\nabla\zeta(T_l(u))$ is square–integrable on $(0,T) \times \mathbf{R}^d$ for any $l > 0$; (ii) for any convex C^2 entropy-entropy flux triple (η, q, r), with η' bounded and $q' = \eta' f'$, $r' = \eta' a$, there exists for any $l > 0$ a nonnegative bounded Radon measure μ_l on $(0,T) \times \mathbf{R}^d$, whose total mass tends to zero as $l \uparrow \infty$, such that

$$(6) \qquad \begin{aligned} &\partial_t \eta(T_l(u)) + \mathrm{div} q(T_l(u)) - \Delta r(T_l(u)) \\ &\quad \le -\eta''(T_l(u)) \left|\nabla \zeta(T_l(u))\right|^2 + \mu_l(t,x) \quad \text{in } \mathcal{D}'((0,T)\times \mathbf{R}^d). \end{aligned}$$

Roughly speaking, (6) expresses the entropy condition satisfied by the truncated function $T_l(u)$. Of course, if u is bounded by M, choosing $l > M$ in (6) yields the usual entropy formulation for u, i.e., a bounded renormalized entropy solution is an entropy solution. However, in contrast to the usual entropy formulation, (6) makes sense also when u is merely L^1 and possibly unbounded. Intuitively the measure μ_l should be supported on $\{|u| = l\}$ and carry information about the behavior of the "energy" on the set where $|u|$ is large. The requirement is that the energy should be small for large values of $|u|$, that is, the total mass of the renormalization measure μ_l should vanish as $l \uparrow \infty$. This is essential for proving uniqueness of a renormalized entropy solution. Being explicit, the existence proof reveals that $\mu_l((0,T)\times \mathbf{R}^d) \le \int_{\{|u_0|>l\}} |u_0| \, dx \to 0$ as $l \uparrow \infty$.

We prove existence of a renormalized entropy solution to (1) using an approximation procedure based on artificial viscosity [20] and bounded data. We derive a priori estimates and pass to the limit in the approximations.

Uniqueness of renormalized entropy solutions is proved by adapting the doubling of variables device due to Kružkov [15]. In the first order case, the uniqueness proof of Kružkov depends crucially on the fact that

$$\nabla_x \Phi(x-y) + \nabla_y \Phi(x-y) = 0, \qquad \Phi \text{ smooth function on } \mathbf{R}^d,$$

which allows for a cancellation of certain singular terms. The proof herein for the second order case relies in addition crucially on the following identity involving the Hessian matrices of $\Phi(x-y)$:

$$\nabla_{xx}\Phi(x-y) + 2\nabla_{xy}\Phi(x-y) + \nabla_{yy}\Phi(x-y) = 0,$$

which, when used together with the parabolic dissipation terms (like the one found in (6)), allows for a cancellation of certain singular terms involving the second order operator in (1). Compared to [9], our uniqueness proof is new even in the case of bounded entropy solutions.

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Perturbations of gradient flow / real Ginzburg-Landau systems

JOHN NORBURY

Suppose we are given the (smooth or C^1) functions $k(x), f(x), g(x)$ which are strictly positive in the closure of the bounded connected domain Ω, and we are given the constants $\epsilon > 0$ and $\alpha \in (0,1)$. Then we consider the parabolic system of partial differential equations (henceforth PDEs)

$$(P) \qquad \begin{cases} \epsilon\frac{\partial u}{\partial t} = \frac{\epsilon^2}{2}\text{div}(k\nabla u) + f^2u[g^2 - u^2 - \alpha v^2] = -\epsilon\frac{\delta E_\epsilon}{\delta u}, \\ \epsilon\frac{\partial v}{\partial t} = \frac{\epsilon^2}{2}\text{div}(k\nabla v) + f^2v[g^2 - v^2 - \alpha u^2] = -\epsilon\frac{\delta E_\epsilon}{\delta v}, \end{cases}$$

for $x \in \Omega$, where

$$E_\epsilon(u,v) = \int_\Omega \frac{\epsilon}{4}(|\nabla u|^2 + |\nabla v|^2) + \frac{1}{\epsilon}W(x,u,v)dx$$

for $u, v \in H^1(\Omega)$, with u, v satisfying homogeneous Neumann boundary conditions $\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0$ on $\partial\Omega$ (which is sufficiently smooth, say C^2, for the derivatives to exist). We are interested in the longtime behaviour of the solutions $u(.,t), v(.,t)$ of such systems, when ϵ is small, and in particular the solutions with changing signs in Ω. Thus we define

$$\mathcal{A}_1 = \mathcal{A}_1(\hat{u}) = \{x : \hat{u}(x) = g(x)/\sqrt{1+\alpha}\},\ \mathcal{A}_2 = \mathcal{A}_2(\hat{v}) = \{x : \hat{v}(x) = g(x)/\sqrt{1+\alpha}\}$$

and describe the curves $\mathcal{C}_i := \partial\mathcal{A}_i$ (i=1,2) as the nodal curves of the solution. These curves $\mathcal{C}_i$ define the pattern of the solution because they separate Ω into subdomains $\mathcal{A}_i(\epsilon), \Omega\setminus\overline{\mathcal{A}_i(\epsilon)} =: \mathcal{A}_i^c(\epsilon)$ such that, if $\mathcal{A}_i(\epsilon) \to \mathcal{A}_i(0)$ as $\epsilon \to 0$, then $\frac{u}{g\sqrt{1+\alpha}}, \frac{v}{g\sqrt{1+\alpha}} \to \pm 1$ for $x \in \mathcal{A}_i(0)$ or for $x \in \mathcal{A}_i(0)^c := \Omega \setminus \overline{\mathcal{A}_i(0)}$. See Figures 6 and 7. The

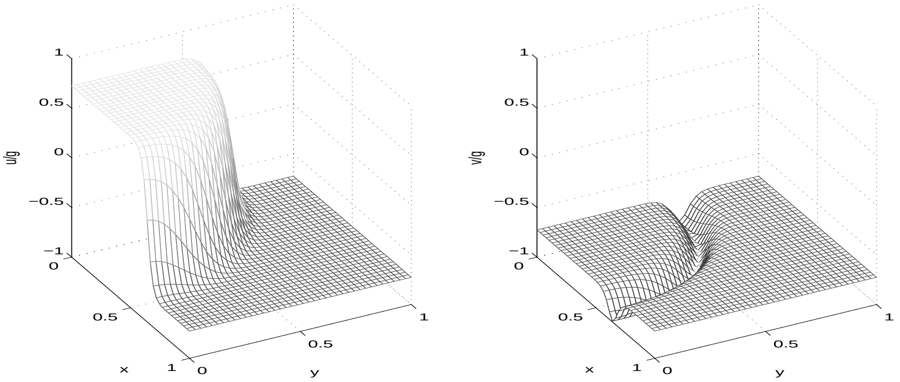

FIGURE 6. Steady solutions to problem (P), with $\Omega = (0,1) \times (0,1)$, $\alpha = 0.9$, $\epsilon = \frac{1}{200}$, $f \equiv k \equiv 1$, and $g(x,y) = \begin{cases} 1 - 3\cosh(\pi r)^2 \exp(-\frac{1}{1-r})\exp(-\frac{1}{r}) & \text{if } r := \sqrt{x^2+y^2} \leq 1; \\ 1 & \text{otherwise.} \end{cases}$
(Steady limit of a numerical simulation of the time-dependent gradient system showing that these local minimisers act as stable attractors for a wide range of initial data with appropriate sign changes.)

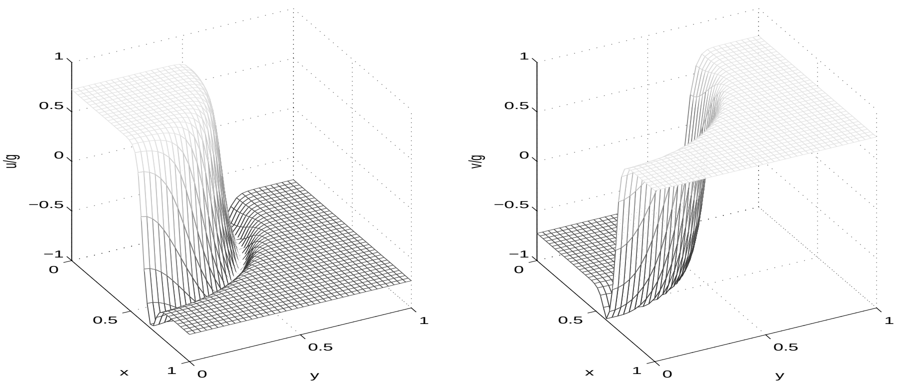

FIGURE 7. Another (stable) solution (with two interfaces) to the same problem as in Figure 6.

talk described the Γ-convergence limit $E_0(u,v)$ of the functional $E_\epsilon(u,v)$ extended to

$u, v \in L^1(\Omega)$ by defining $E_\epsilon(u,v) = \infty$ for $u, v \in L^1(\Omega) \setminus H^1(\Omega)$. In fact, as a simple one-dimensional example for the steady solutions of the following nonlinearly forced heat equation in one space dimension with boundary conditions $u_x(-1) = 0 = u_x(1)$, shows, $u(x)$ tends to a bounded discontinuous limit as $\epsilon \to 0$, and so for appropriate initial data (which of necessity must change sign in Ω), the corresponding time dependent problem

$$\epsilon u_t = \frac{\epsilon^2}{2} u_{xx} + u[g^2 - u^2]$$

will have attracting steady states which become discontinuous in Ω in the limit $\epsilon \to 0$. Hence this example (embedded using the appropriate symmetrical domain Ω) shows that in general the Γ-limit $E_0(u,v)$ cannot be defined on elements u, v in the (Banach) space $H^1(\Omega)$, but may be bounded in the Banach space of functions of bounded variation $BV(\Omega)$, where $\|w\|_{BV} := \|w\|_{L^1(\Omega)} + \int_\Omega |Dw|$

$$\text{and} \qquad \int_\Omega |Dw| := \sup\left\{ \int_\Omega w \operatorname{div} \phi \, dx \; : \phi \in C_0^1(\Omega, \mathbb{R}^2), |\phi| \leq 1 \right\} < \infty.$$

In fact we can show that the $\epsilon \to 0$ limits of solutions u, v of our problem always lie in the subspace $SBV(\Omega)$ (where $H^1(\Omega) \subset SBV(\Omega) \subset BV(\Omega) \subset L^1(\Omega)$), the subspace of special functions of bounded variation that possess no Cantor part in the measure valued derivative (in other words, the singular measure in the generalised derivative always consists of a bounded jump $u_-(x) < u_+(x)$ for $x \in J(u)$, the jump set of Hausdorff dimension one in Ω; note that this jump set has a generalised normal ν and belongs to a generalised curve in the geometric measure theory sense, see [1], [2]).

The key result of Girardet and Norbury [3], that

$$E_0(u,v) = K(\alpha) \left\{ \int_\Omega \sqrt{k} f g^3 |D\chi_{\mathcal{A}_1}| + \int_\Omega \sqrt{k} f g^3 |D\chi_{\mathcal{A}_2}| \right\}$$

for $u, v \in SBV(\Omega)$, is interpreted as an equation for extremal geodesics $\mathcal{C}(0) \subset \Omega$ that may act as stable attractors for our time dependent dynamical system problem when $\mathcal{C}(0)$ are isolated local minimisers of $E_0(u,v)$ in $SBV(\Omega) \times SBV(\Omega)$. These (extremal) geodesics for the domain Ω are calculated from the equation

$$\kappa(x) = -\frac{\partial}{\partial n} \ln h(x)$$

for $x \in \mathcal{C}(0) \subset \Omega$, where $\kappa(x)$ is the (Gaussian) mean curvature of $\mathcal{C}(0)$ at the point x and $\frac{\partial}{\partial n}$ is the normal derivative to $\mathcal{C}(0)$ (given the direction of an increasing arc length parameter as the direction of the tangent, and with the normal then making the usual sense to the tangent), and where the metric $h(x) := \sqrt{k(x)} f(x) g(x)^3$ for $x \in \Omega$, is smooth $(C^1(\bar{\Omega}))$ and strictly positive in $\bar{\Omega}$. Either a geodesic intersects $\partial\Omega$, or otherwise extremal geodesics are periodic, and (at least) C^1 in the arc length parameter (note that geodesics may intersect, including self-intersections). New features of problem (P) that were described include:

(a) "Ridges" in one solution component exist which act as indicators or markers of interfaces and their nodal curves in the other solution component (these ridges are proportional in height to α and persist as $\epsilon \to 0$ in the component (say u) that keeps the same sign when the other component (say v) changes sign, where both the centrelines of such a ridge and such an interface converge to $\mathcal{C}(0)$ as $\epsilon \to 0$).

(b) Stable "double interface" solutions exist whenever a stable single interface solution exists (the double interface is of the type where there is a single interface in each component, and the distance between the single interfaces vanishes as $\epsilon \to 0$ but does not remain uniformly $\mathcal{O}(\epsilon)$).

These new features appear in the solutions of the coupled ordinary differential equation problem

$$0 = \frac{1}{2}u_{xx} + u[1 - u^2 - \alpha v^2], \qquad 0 = \frac{1}{2}v_{xx} + v[1 - v^2 - \alpha u^2],$$

for $-\infty < x < \infty$, where $u_x, v_x \to 0$ as $|x| \to \infty$; here $u \to \pm(1+\alpha)^{-\frac{1}{2}}$ as $x \to \pm\infty$, while either $v \to (1+\alpha)^{-\frac{1}{2}}$ as $x \to \pm\infty$ in the single interface case, or $v \to \mp(1+\alpha)^{-\frac{1}{2}}$ as $x \to \pm\infty$ in the double interface case (see Girardet and Norbury [4] where this ordinary differential equation problem is further analysed). This problem models the key behaviour in Problem (P) near interfaces when x is a variable measuring stretched (in ϵ) distance normal to the interface, and the u, v are scaled by g.

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Well-Posedness of Free Surface Problems In 2D Fluids

David Ambrose

The main subject of this talk is my recent proof of a long-standing conjecture in fluid dynamics: that the motion of a vortex sheet subject to surface tension is well-posed (for a short time). The method employed was strongly related to numerical methods developed by T. Hou, J. Lowengrub, and M. Shelley [4]. In particular, both analysis and computation of the problem become possible when the problem is reformulated using natural variables and convenient parameterizations. This will be described in more detail below. Even more recently, Nader Masmoudi and I have extended the analytical method to provide a new proof of the well-posedness of two-dimensional water waves without surface tension.

The proofs of well-posedness for both the water wave and for the vortex sheet with surface tension allow a wide class of initial data to be used. The position of the interface and the initial vorticity (which is concentrated on the interface) are taken in Sobolev spaces, and the interface may be of multiple heights. Also, there is no restriction on the size of the initial data; the only requirement is that a natural non-self-intersection condition must be met.

The vortex sheet is the interface between two incompressible, irrotational, inviscid fluids flowing past each other. It is well known that without surface tension, the vortex sheet is ill-posed. It had long been believed that when surface tension is accounted for in the evolution equations, the initial value problem would become well-posed.

At each time, the sheet can be viewed as a curve in the complex plane. The curve, z, is parameterized by a spatial variable, α, and by time, t. The classical vortex sheet evolves according to the Birkhoff-Rott integral,

$$z_t^*(\alpha, t) = \frac{1}{2\pi i}\mathrm{PV}\int_{-\infty}^{\infty} \frac{\gamma(\alpha')}{z(\alpha,t) - z(\alpha',t)}\, d\alpha'. \tag{1}$$

The $*$ denotes complex conjugation; γ is the vortex sheet strength. Notice that γ is not a function of time. This problem has been studied for many years and has been found to be ill-posed. In particular, it exhibits the well-known Kelvin-Helmholtz instability: in the linearization of the evolution equations about equilibrium, Fourier modes with high wave numbers grow without bound. Equation (1) neglects the effect of surface tension at the interface. Surface tension is a restoring force, and when surface tension is accounted for in the equations of motion, Fourier modes of high wave number remain bounded in the linearization. Taking this further, Beale, Hou, and Lowengrub demonstrated that even far from equilibrium, surface tension makes the linearized equations well-posed [5]. For these reasons, it had been conjectured that surface tension makes the full problem well-posed.

The HLS formulation has two important components: first, they compute dependent variables which are naturally related to the surface tension. In particular, surface tension enters the evolution equations in the form $\gamma_t = \frac{1}{\mathrm{We}}\kappa_\alpha$, where κ is the curvature of the vortex sheet and We is the Weber number. The Weber number is a dimensionless parameter that is inversely proportional to the surface tension; the case without surface tension corresponds to $\mathrm{We} = \infty$. (Recall that without surface tension, $\gamma_t = 0$.) To simplify this curvature term in the evolution equations, Hou, Lowengrub, and Shelley described the curve by its tangent angle and arclength rather than by the Cartesian variable, z. The notation s will be used for arclength and θ will be used for the tangent angle the curve forms with the horizontal. The strength of this choice of variables lies in the close relationship between curvature and the tangent angle, $\kappa = \theta_\alpha / s_\alpha$.

Second, HLS added an extra tangential velocity to the evolution equation for the vortex sheet. This does not change the shape of the vortex sheet; rather, it only reparameterizes it. The most convenient choice will make the curve always be parameterized by arclength. Since the evolution equation for s_α is

$$s_{\alpha t} = T_\alpha - \theta_\alpha U, \tag{2}$$

where T and U are the tangential and normal velocities of z, T can be chosen to essentially eliminate s_α from the system. That is, setting $s_{\alpha t}$ equal to a function of time yields an equation for T. This essentially reduces the problem by one dependent variable. Rather than evolving $s_\alpha(\alpha, t)$, it is only necessary to keep track of $L(t)$, the length of one period of the sheet.

The choice of tangential velocity also changes the evolution equation for γ. These new terms introduced in the γ_t equation are of a lower order than the term which comes from surface tension.

The main tool in the proof of well-posedness of vortex sheets with surface tension is an energy estimate. This energy estimate is performed after much rewriting of the evolution equations. The equations can be stated as

$$\theta_t = \frac{2\pi^2}{L^2} H(\gamma_\alpha) + P, \tag{3}$$

$$\gamma_t = \frac{2\pi}{L\mathrm{We}} \theta_{\alpha\alpha} + \frac{2\pi^2}{L^2} \gamma H(\gamma\theta_\alpha) + \widetilde{Q}. \tag{4}$$

The Hilbert transform is denoted by H. Since L is a function of t only, notice that the evolution equations are effectively semilinear. After performing the energy estimate, standard methods can be applied to prove well-posedness. The energy functional is related the H^s Sobolev norm of θ and the $H^{s-1/2}$ Sobolev norm of γ.

Also, the analysis described above proves well-posedness in the case where the upper and lower fluids have different densities. With a density difference, the γ_t equation has additional terms, although none of them are significant. (It is worth noting that in the two-density case, the equation for γ_t is actually an integral equation for γ_t; it was proven in [2] that this integral equation can be solved.) A particular case of the two-density problem is when the upper fluid has density equal to zero; this is the water wave. Thus, the work described above establishes well-posedness of the two-dimensional water wave with surface tension.

Without surface tension, well-posedness of the full water wave problem was demonstrated by Wu in [6]. The proof, however, requires significant use of complex analysis (the Riemann mapping theorem in particular). In [3], Nader Masmoudi and I have given a simpler proof of well-posedness for the two-dimensional irrotational water wave. The method resembles the method of [1], but there are important differences. In particular, the variable γ is insufficient in the water wave case. An appropriate new variable is δ, the difference between the Lagrangian tangential velocity and the special tangential velocity: $\delta = \mathbf{W} \cdot \hat{\mathbf{t}} + \frac{\gamma}{2s_\alpha} - T$. Frequently the derivative δ_α is more useful than δ itself. The evolution equation for δ_α can be written

$$\delta_{\alpha t} = -c\theta_\alpha + \psi, \tag{5}$$

where ψ is a collection of terms which are easy to deal with when performing energy estimates. Here, $c = c(\alpha, t)$ is defined by $c = -\nabla p \cdot \hat{\mathbf{n}}$. A necessary condition for well-posedness is that $c(\alpha, t) > 0$. This is a generalization of a condition of G. I. Taylor.

In terms of δ instead of γ, the evolution equation for θ can be written

$$\theta_t = \frac{2\pi}{L} H(\delta_\alpha) + \phi. \tag{6}$$

Here, ϕ is a collection of terms which can be handled routinely in the energy estimates.

Energy estimates for the system (5), (6) can be made with $\delta_\alpha \in H^{s-1/2}$ and $\theta \in H^s$, for s large enough. Similar estimates hold when surface tension is accounted for. It is then found that solutions to the water wave problem without surface tension are the limit of solutions to the water wave problem with surface tension as surface tension goes to zero.

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Stability of Periodic Peakons

JONATAN LENELLS

Abstract. The peakons are peaked traveling wave solutions of a nonlinear integrable equation modeling shallow water waves. We give a simple proof of their stability.

AMS SUBJECT CLASSIFICATION (2000): 35Q35, 37K45.
KEYWORDS: Water waves, Peakons, Stability.

INTRODUCTION

The Camassa-Holm equation

$$u_t - u_{txx} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \tag{0.1}$$

arises as a model for the unidirectional propagation of shallow water waves over a flat bottom, $u(x,t)$ representing the water's free surface in non-dimensional variables. We are concerned with periodic solutions of (0.1), i.e. $u : \mathbb{S} \times [0,T) \to \mathbb{R}$ where $\mathbb{S}$ denotes the unit circle and $T > 0$ is the maximal existence time of the solution. Equation (0.1) was first obtained [6] as an abstract bi-Hamiltonian equation with infinitely many conservation laws and was subsequently derived from physical principles [2]. Equation (0.1) is a re-expression of the geodesic flow in the group of compressible diffeomorphisms of the circle [7], just like the Euler equation is an expression of the geodesic flow in the group of incompressible diffeomorphisms of the torus [1]. This geometric interpretation leads to a proof that equation (0.1) satisfies the Least Action Principle [3]: a state of the system is transformed to another nearby state through a uniquely determined flow that minimizes the energy. For a large class of initial data, equation (0.1) is an infinite-dimensional completely integrable Hamiltonian system: by means of an isospectral problem one can

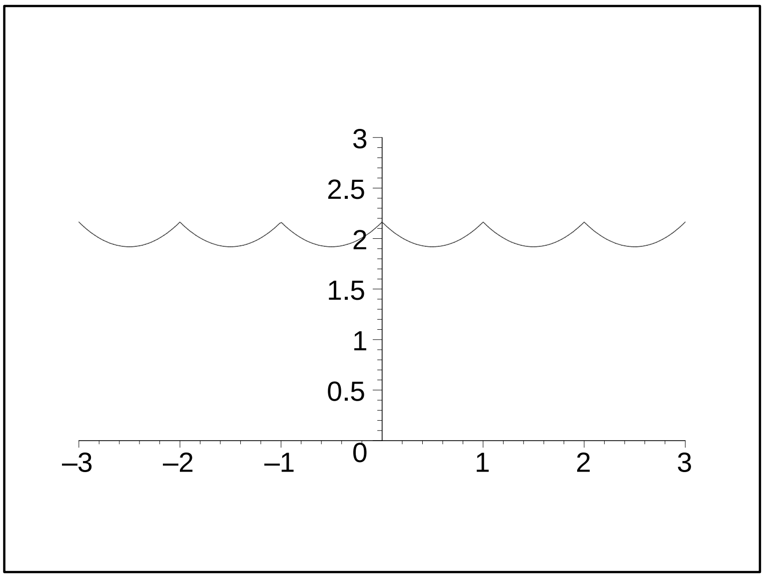

FIGURE 8. The peakon for $c = \frac{\cosh(1/2)}{\sinh(1/2)}$.

convert the equation into an infinite sequence of linear ordinary differential equations which can be trivially integrated [4].

Equation (0.1) has the periodic traveling solution

$$u(x,t) = \frac{c\varphi(x-ct)}{M_\varphi}, \quad c \in \mathbb{R},$$

where $\varphi(x)$ is given for $x \in [0,1]$ by

$$\varphi(x) = \frac{\cosh(1/2 - x)}{\sinh(1/2)}$$

and extends periodically to the real line, and

$$M_\varphi = \max_{x \in S}\{\varphi(x)\} = \frac{\cosh(1/2)}{\sinh(1/2)}.$$

Because of their shape (they are smooth except for a peak at their crest, see Figure 1) these solutions are called (periodic) peakons. Note that the height of the peakon is equal to its speed. Equation (0.1) can be rewritten in conservation form as

$$u_t + \frac{1}{2}\left(u^2 + \varphi * [u^2 + \frac{1}{2}u_x^2]\right)_x = 0. \tag{0.2}$$

This is the exact meaning in which the peakons are solutions.

Numerical simulations suggest that the sizes and velocities of the peakons do not change as a result of collision so that these patterns are expected to be stable. Moreover, for the peakons to be physically observable it is necessary that their shape remains approximately the same as time evolves. Therefore the stability of the peakons is of great interest. We prove the following:

Theorem *The periodic peakons are stable.*

Outline of Proof. Equation (0.1) has the conservation laws

$$(0.3)\quad H_0[u]=\int_{\mathbb{S}} u\,dx,\quad H_1[u]=\frac{1}{2}\int_{\mathbb{S}}(u^2+u_x^2)dx,\quad H_2[u]=\frac{1}{2}\int_{\mathbb{S}}(u^3+uu_x^2)dx.$$

To each solution $u(x,t)$ we associate a function $F_u(M,m)$ of two real variables (M,m) depending only on the three conservation laws H_0, H_1, H_2. Since H_0, H_1, H_2 are conserved quantities, F_u does not depend on time. If we let $M_{u(t)}=\max_{x\in S}\{u(x,t)\}$ and $m_{u(t)}=\min_{x\in S}\{u(x,t)\}$ be the maximum, respectively the minimum of u at the time t, it turns out that

$$(0.4)\qquad F_u(M_{u(t)},m_{u(t)})\geq 0,\quad t\in[0,T).$$

Moreover, for the peakon we have $F_\varphi(M,m)\leq 0$ with equality only at the point (M_φ,m_φ) (see Figure 2). If u is a solution starting close to φ, the conserved quantities $H_i[u]$ are close to $H_i[\varphi]$, $i=0,1,2$, and hence F_u is a small perturbation of F_φ. Therefore, the set where $F_u\geq 0$ is contained in a small neighborhood of (M_φ,m_φ). We conclude by (0.4) that $(M_{u(t)},m_{u(t)})$ stays close to (M_φ,m_φ) for all times. The proof is completed by showing that if the maximum of u stays close to the maximum of the peakon, then the shape of the whole wave remains close to that of the peakon. □

The proof is inspired by [5] where the case of peaked solitary waves of (0.1) is considered. The approach taken here is similar but there are differences. For example, in the case of the peaked solitary waves [5] a polynomial in M plays the role of our function $F_u(M,m)$. The variable m enters in the periodic case because of non-zero boundary conditions.

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Well-posedness of KdV on $H^{-1}(\mathbb{T})$

THOMAS KAPPELER
(joint work with Peter Topalov)

Let us consider the Initial Value Problem (IVP) for the Korteweg-deVries equation on the circle

$$v_t = -v_{xxx} + 6vv_x \qquad t \in \mathbb{R},\ x \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$$
$$v\big|_{t=0} = q \in H^\alpha(\mathbb{T}).$$

This problem has been studied extensively. In particular it is known that for $q \in C^\infty(\mathbb{T})$, the (IVP) admits a unique solution $\mathcal{S}(t,q)$ which exists for all times (see [BS]). Our aim is to solve the (IVP) for very rough initial data such as distributions in the Sobolev space $H^{-1}(\mathbb{T})$.

We say that a continuous curve $\gamma : [T_1, T_2] \to H^\alpha(\mathbb{T})$ with $T_1 < 0 < T_2, \gamma(0) = q$ and $\alpha \in \mathbb{R}$ is a solution of (IVP) if for any $T_1 < t < T_2$ and for any sequence $(q_k)_{k\geq 1} \subseteq C^\infty(\mathbb{T})$ with $q = \lim_{k\to\infty} q_k$ in $H^\alpha(\mathbb{T})$, the solutions $\mathcal{S}(\cdot, q_k)$ have the property that $\gamma(t) = \lim_{k\to\infty} \mathcal{S}(t, q_k)$ in $H^\alpha(\mathbb{T})$. It then follows from the definition of a solution of (IVP) that it is unique whenever it exists. If the solution of (IVP) exists, we denote it by $\mathcal{S}(t,q)$.

The above (IVP) is said to be globally [uniformly] C^0-wellposed on $H^\alpha(\mathbb{T})$ if for any $q \in H^\alpha(\mathbb{T})$ the solution $\mathcal{S}(t,q)$ exists globally in time and the solution map $\mathcal{S}$ is continuous [uniformly continuous on bounded sets] as a map $\mathcal{S} : H^\alpha(\mathbb{T}) \to C^0(\mathbb{R}, H^\alpha(\mathbb{T}))$.

Theorem 1. ([KT1]) KdV is globally C^0-wellposed on $H^\alpha(\mathbb{T})$ for any $-1 \leq \alpha \leq 0$.

Remarks: (1) Theorem 1 improves in particular on earlier results of [Bou1], [Bou2], [KPV], [CKSTT]. Using earlier results, it is proved in [CKSTT] that KdV is globally uniformly C^0-wellposed on $H_0^\alpha(\mathbb{T})$ for any $\alpha \geq -1/2$.

(2) In [CCT] it is shown that KdV is *not* uniformly C^0-wellposed on $H_0^\alpha(\mathbb{T})$ for $-2 < \alpha < -1/2$ where $H_0^\alpha(\mathbb{T}) = \{q \in H^\alpha(\mathbb{T}) \mid \int_{\mathbb{T}} q = 0\}$. See also [Bou2].

The following theorem states that well known features [MT] of solutions of (IVP) for smooth initial data continue to hold for rough initial data.

Theorem 2. ([KT1]) For any $q \in H^\alpha(\mathbb{T})$ with $-1 \leq \alpha \leq 0$, the solution of (IVP) has the following properties:

(i) the orbit $t \mapsto \mathcal{S}(t,q)$ is relatively compact.
(ii) $t \mapsto \mathcal{S}(t,q)$ is almost periodic.

Theorem 1 and Theorem 2 can be applied to obtain corresponding results for the IVP of the modified KdV (mKdV)

$$u_t = -u_{xxx} + 6u^2 u_x \qquad t \in \mathbb{R},\ x \in \mathbb{T}$$

$$u\big|_{t=0} = r \in H^\alpha(\mathbb{T}).$$

Theorem 3. ([KT2]) mKdV is globally C^0-wellposed on $H^\alpha(\mathbb{T})$ for $0 \le \alpha \le 1$.

Remarks: (1) Theorem 3 improves on earlier results of [Bou1], [KPV], [CKSTT]. Using earlier results it is proved in [CKSTT] that mKdV is globally uniformly C^0-wellposed on $H^\alpha(\mathbb{T})$ for any $\alpha \ge 1/2$.

(2) In [CCT] it is shown that mKdV is *not* uniformly C^0-wellposed on $H_0^\alpha(\mathbb{T})$ for $-1 < \alpha < 1/2$. See also [Bou2].

Besides Theorem 1, the main ingredient of the proof of Theorem 3 is the following result on the Miura map, $B : L^2(\mathbb{T}) \to H^{-1}(\mathbb{T}), r \mapsto r_x + r^2$, first introduced by Miura [Mi] and proved to be a Bäcklund transformation, mapping solutions of mKdV to solutions of KdV.

Theorem 4. ([KT2])

(i) For any $\alpha \ge 0$, the Miura map $B : H^\alpha(\mathbb{T}) \to H^{\alpha-1}(\mathbb{T})$ is a global fold.

(ii) Restricted to $H_0^\alpha(\mathbb{T})$, B is a real analytic isomorphism onto the real analytic submanifold $H_0^{\alpha-1}(\mathbb{T}) := \{q \in H^{\alpha-1}(\mathbb{T}) \,|\, \lambda_0(q) = 0\}$ where $\lambda_0(q)$ denotes the lowest eigenvalue in the periodic spectrum of the operator $-d^2/dx^2 + q$.

Remark: Theorem 4 is based on earlier results on the Riccati map [KT3] which used as one of the ingredients estimates on the gaps of the periodic spectrum of impedance operators of [Kor1]. Some of the results in [KT3] have been obtained independently by [Kor2].

The main ingredient in the proof of Theorem 1 is a result on the normal form of the Korteweg-deVries equation considered as an integrable Hamiltonian system. To formulate it, introduce the following model spaces $(\alpha \in \mathbb{R})$

$$h^\alpha := \{(x_k, y_k)_{k\ge1} \,\big|\, x_k, y_k \in \mathbb{R};\ \sum_{k\ge1} k^{2\alpha}(x_k^2 + y_k^2) < \infty\}$$

with the standard Poisson bracket where $\{x_k, y_k\} = 1 = -\{y_k, x_k\}$ and all other brackets between the coordinate functions vanish.

On the space $H_0^\alpha(\mathbb{T}) := \{q = \sum_{k\ne0} \hat{q}_k e^{2\pi ikx} \,\big|\, q \in H^\alpha(\mathbb{T})\}$ we consider the Poisson bracket introduced by Gardner and, independently, by Faddeev and Zakharov

$$\{F, G\} = \int_{\mathbb{T}} \frac{\partial F}{\partial q(x)} \frac{d}{dx} \frac{\partial G}{\partial q(x)} dx.$$

Theorem 5. ([KP], [KMT]) There exists a real analytic diffeomorphism $\Omega : H_0^{-1}(\mathbb{T}) \to h^{-1/2}$ so that

(i) Ω preserves the Poisson bracket;

(ii) for any $-1 \leq \alpha \leq 0$, the restriction Ω_α of Ω to $H_0^\alpha(\mathbb{T})$ is a real analytic isomorphism, $\Omega_\alpha : H_0^\alpha(\mathbb{T}) \to h^{\alpha+1/2}$;

(iii) on $H_0^1(\mathbb{T})$, the KdV Hamiltonian $\mathcal{H}(q) = \int_{\mathbb{T}}(\frac{1}{2}q_x^2 + q^3)dx$, when expressed in the new coordinates $(x_k, y_k)_{k\geq 1}$, is a real analytic function of the actions $I_k := (x_k^2 + y_k^2)/2$ $(k \geq 1)$ alone.

Remark: In [KP] it is shown that $\Omega_0 : L_0^2 \to h^{1/2}$ is a real analytic isomorphism with properties (i) and (iii). Moreover it is proved that for any $\alpha \in \mathbb{N}$, the restriction Ω_α of Ω to $H_0^\alpha(\mathbb{T})$ is a real analytic isomorphism, $\Omega_\alpha : H_0^\alpha(\mathbb{T}) \to h^{\alpha+1/2}$. This result has been extended in [KMT] as formulated in Theorem 5.

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Exact Periodic water waves with Vorticity

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(joint work with Adrian Constantin)

The work presented here is an application of global continuation methods to a classical problem in fluid mechanics. The main tools are Leray-Schauder degree, bifurcation theory, and estimates for elliptic PDEs.

We consider the most classical kind of water wave, traveling at a constant speed $c > 0$. We assume that it is two-dimensional and horizontally periodic with a period L. The water is treated as incompressible and inviscid. We denote the horizontal variable by x and the vertical variable by y. The bottom is assumed to be flat. The surface S has average height $y = 0$. Let S be given by the equation $y = \eta(x - ct)$. Gravity acts on the water with gravitational constant g. The air pressure is assumed to be a constant P_{atm} and it is assumed that there is no surface tension. Let (u, v) denote the velocity, ψ denote the stream function and ω the vorticity. Then $\omega = \gamma(\psi)$ for some function γ.

The special case of irrotational flow, when $\omega = 0$, has been studied much more than the general case because then ψ is a harmonic function and the techniques of complex analysis are readily available. However, we want to focus on the case of general vorticity.

It follows from the preceding equations that the relative mass flux

$$p_0 = \int_{-d}^{\eta(x)} \{u(x, y) - c\} dy$$

is independent of x. We will be looking for waves with $u < c$ and therefore $p_0 < 0$.

Theorem 1 (Main Theorem)**.** *Let $L > 0, c > 0, p_0 < 0$. Let these quantities satisfy Condition A given below. Then there exist traveling waves (u, v, η) of period L, flux p_0 and speed c, with $u < c$, which are symmetric around each crest and trough. In fact there exists a connected set $\mathcal{C}$ of such waves in the space $C2 \times C2 \times C3$ such that*

(i) $\mathcal{C}$ contains a trivial flow with $\eta \equiv 0$ (that is, a flat surface), and

(ii) $\mathcal{C}$ contains waves for which $\max u \nearrow c$ (that is, stagnation).

Condition A. *Let $\Gamma'(p) = \gamma(-p)$ with $\Gamma(0) = 0$. Denote $a(p) = \sqrt{\lambda + 2\Gamma(p)}$ defined for $\lambda > -2\Gamma_{min}$. Condition A requires that for some λ the Sturm-Liouville problem*

$$-(a3M_p)_p = \mu a M, \quad M(p_0) = 0, \quad a^3 M_p(0) = gM(0)$$

has an eigenvalue $\mu \leq -1$.

This condition is necessary for the validity of the theorem. We will construct the continuum $\mathcal{C}$ by bifurcation from the curve of trivial solutions. Condition A is required for the existence of a local bifurcation curve.

The proof is based in part on the following ingredients: (1) a transformation due to Choquet-Bruhat that fixes the free boundary, (2) a local bifurcation argument using the Crandall-Rabinowitz theorem, (3) a global bifurcation argument of Rabinowitz type using the Healey-Simpson degree, (4) a nodal characterization of the solutions using the Hopf and Serrin maximum principles, and (5) regularity theorems of Schauder type due to Lieberman and Trudinger for fully nonlinear elliptic problems.

On Stokes's extreme wave

LUDWIG EDWARD FRAENKEL

Stokes conjectured in 1880 that (in the absence of surface tension and viscosity) the 'highest' gravity wave on water

(i) is distinguished by sharp crests of included angle $2\pi/3$;
(ii) has a profile (by which we mean the free upper boundary of the water) that is convex between successive crests.

Part (i) of this conjecture was proved in 1982 in two quite different ways. In England, Amick, Fraenkel and Toland used the integral equation of Nekrasov and real-variable methods for functions of one variable. In Novosibirsk, Plotnikov used complex-variable methods and an extension of a certain function beyond its domain in the plane of the complex potential. (This was an inspired sharpening for a particular case of a general construction due to H. Lewy.)
Part (ii) of the conjecture has been proved recently by Plotnikov and Toland (who have been collaborators since 1997). The proof uses complex-variable methods of the kind initiated by Plotnikov; it is a tour de force, but far from simple.

The present talk describes two unsuccessful attempts to obtain a relatively simple existence proof for the extreme wave by means of the Nekrasov equation, in the hope that such a proof might yield both parts of the Stokes conjecture more or less directly.

For periodic gravity waves on water of infinite depth, the Nekrasov equation is

$$\phi(s) = (T_\nu \phi)(s) := \frac{1}{3}\int_0^\pi K(s,t)\,\frac{\sin\phi(t)}{\nu + \int_0^t \sin\phi}\,dt, \quad 0 < s \le \pi,$$

where

$$K(s,t) := \frac{1}{\pi}\log\frac{\tan\frac{1}{2}s + \tan\frac{1}{2}t}{|\tan\frac{1}{2}s - \tan\frac{1}{2}t|},$$

and where $\tan\phi(s) := Y'(x) \ge 0$ is the slope of half a wave-length of the free boundary $\{(x, Y(x)) \mid x \in \mathbb{R}\}$. The points $s = \pi$ and $s = 0$ correspond to a trough and a crest, respectively. The parameter $\nu \in \left[0, \frac{1}{3}\right)$ and is such that $\frac{1}{3} - \nu$ is small for waves of small amplitude, while $\nu = 0$ for the extreme (or 'highest') wave. In 1978, Toland proved the existence of a suitable solution for $\nu = 0$ by considering a sequence $\left(\phi_{\nu(n)}\right)_{n=1}^\infty$ of solutions for which $\nu(n) \downarrow 0$ as $n \to \infty$.

My first attempt involves the sequence $(\phi_n)_{n=0}^\infty$ of functions defined by $\sin\phi_0(s) := \frac{1}{2}\cos\frac{s}{2}$ $(0 \le s \le \pi)$ and $\phi_{n+1} := T_0\phi_n$. The functions ϕ_1 and ϕ_2 are known exactly; ϕ_3 and ϕ_4 are described by formulae which result partly from fitting a trigonometric series to numerical values of the smooth part of $\sin\phi_n(t)/\int_0^t \sin\phi_n$ for $n = 2$ and 3. Graphs of ϕ_0 to ϕ_4 suggest rapid convergence. The leading four terms of $\phi_n(s)$ for $s \downarrow 0$ are known for every n and form a part of the formulae for ϕ_0 to ϕ_4.
However, I have failed to prove convergence.

In the equation $\phi = T_0\phi$ for the extreme wave, let

$$(\mathcal{N}\phi)(s) := \frac{\sin\phi(s)}{\int_0^s \sin\phi}$$

and

$$(\mathcal{K}f)(s) := \int_0^\pi K(s,t)\, f(t)\, dt \qquad (0 < s \le \pi),$$

so that $T_0\phi \equiv \frac{1}{3}\mathcal{K} \circ \mathcal{N}\phi$. Perhaps the main difficulty of the problem is that $\mathcal{N}\phi$ is not a monotonic function of ϕ (under the usual partial ordering of continuous functions on $(0,\pi]$). However, the inverse not only exists but is an increasing function of $\mathcal{N}\phi$; in fact,

$$\sin\psi(s) = \sin\frac{s}{2}\ \mathcal{N}\psi(s)\ \exp\int_0^s \left\{\mathcal{N}\psi(t) - \frac{1}{2}\cot\frac{t}{2}\right\} dt$$

whenever $\mathcal{N}\psi(t) \sim 1/t$ as $t \downarrow 0$, $\mathcal{N}\psi \in C(0,\pi]$ and $\mathcal{N}\psi(s) \ge 0$.

Accordingly, my second attempt has been to pursue $f := \mathcal{N}\phi$, rather than ϕ, by means of the new equation $f = Af$, where

$$\begin{aligned}(Af)(s):\ &= \frac{\sin(\frac{1}{3}\mathcal{K}f)(s)}{\sin\frac{s}{2}}\quad \exp\textstyle\int_0^s\{f_0 - f\}, \qquad f_0(t) := \mathcal{N}\phi_0(t) = \frac{1}{2}\cot\frac{t}{2},\\ &= \frac{\sin(T_0\phi)(s)}{\int_0^s \sin\phi}\qquad \text{if}\quad \phi := \mathcal{N}^{-1}f.\end{aligned}$$

An encouraging property of this equation is that its linearization about f_0 is solvable, as follows. If we set $f = f_0 + h$, then, formally,

$$f = Af \iff h - A'(f_0)h = Af_0 - f_0 + O(h^2).$$

Theorem. *The equation*

$$h - A'(f_0)h = g \quad \textit{in} \quad L_2 := L_2(0,\pi)$$

has a unique solution satisfying

$$\| h \|_{L_2} \le \left(\frac{7}{9} - \frac{3}{4}\log\frac{4}{3}\right)^{-1} \| g \|_{L_2} .$$

Uniqueness issues on permanent progressive water-waves

HISASHI OKAMOTO

(joint work with Kenta Kobayashi)

Abstract. We consider two-dimensional water-waves of permanent shape with constant propagation speed. Two theorems concerning the uniqueness of certain solutions are reported. Uniqueness of Crapper's pure capillary waves is proved under a positivity assumption. The proof is based on the theory of positive operators. Also proved is the uniqueness of the gravity waves of mode one. This is done by a combination of new inequalities and numerical verification algorithm.

Keywords. Crapper's wave, gravity waves, uniqueness, positivity, the Perron-Frobenius theory, verified numerics.

SUMMARY

We consider progressive waves of permanent shape on 2D irrotational flow of incompressible inviscid fluid. For the sake of simplicity, we consider only those fluid flows whose depth are infinite. We show that, under a positivity assumption, the pure capillary waves of Crapper are unique. Also, the positive gravity waves are shown to be unique.

Specifically we consider a solution θ of

$$q\frac{\mathrm{d}\theta}{\mathrm{d}\sigma} = -\sinh(H\theta) \qquad (-\pi \leq \sigma \leq \pi) \tag{1}$$

such that θ is 2π-periodic and satisfies $\theta(-\sigma) = -\theta(\sigma)$. For its meaning, see [5]. In 1957, G.D. Crapper found a family of solutions of (1), which are written, in our context, as follows:

$$q = \frac{1 + A2}{1 - A2}, \qquad \theta(\sigma) = -2\arctan\left(\frac{2A\sin\sigma}{1 - A2}\right).$$

A natural question would be: Does the differential equation (1) has a solution other than Crapper's waves?

Our results is:

Theorem 1. *Suppose that a solution of* (1) *satisfies* $\theta(-\sigma) = -\theta(\sigma)$ *and either the following* **A1** *or* **A2***.*

A1: $0 \leq \theta(\sigma) \leq \pi$ *everywhere in* $0 \leq \sigma \leq \pi$*;*
A2: $\frac{\mathrm{d}\tau}{\mathrm{d}\sigma}(\sigma) \geq 0$ *everywhere in* $0 \leq \sigma \leq \pi$*.*

Then it is one of Crapper's solutions of mode one.

The proof of this theorem depends crucially on [7]. See [6].

We now move on to a uniqueness theorem on the gravity waves, which has been recently obtained by the second author. Now the assumption is that the surface tension is neglected and only the gravity acts. In this case the solutions are obtained by solving the following integral equation, called Nekrasov's equation:

$$\theta(\sigma) = \frac{1}{3\pi}\int_0^\pi \log\left|\frac{\sin\frac{\sigma+s}{2}}{\sin\frac{\sigma-s}{2}}\right| \frac{\mu\sin\theta(s)}{1 + \mu\int_0^s \sin(\theta(u))\mathrm{d}u}\mathrm{d}s. \tag{2}$$

Here μ is a new parameter related to the gravity acceleration.

The equation (2) has a rather long history but the structure of the solutions had long been unclear except for those solutions of small amplitude. See [5]. The first satisfactory answer was given by [3] as in the following form:

Theorem 2 (Keady & Norbury, '78)**.** *For all* $3 < \mu < \infty$*, there exists at least one non-trivial solution satisfying* $0 \leq \theta \leq \pi/2$*.*

It is known that there exist solutions which change sign in $0 \le \sigma \le \pi$. It is also known that secondary bifurcations exist along such solutions. Therefore uniqueness does not hold among solutions of different signs. However, no secondary bifurcation seems to exist along a positive solutions, and we expect uniqueness for positive solutions. The second author proved in [4] the following

Theorem 3. *For all $3 < \mu \le 40.0$, there exists at most one non-trivial solution satisfying $0 \le \theta \le \pi$.*

The proof in [4] uses the validated numerics or “interval analysis”, which gives us exact (i.e., including round-off error) bound for numerical computations.

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On the spectral problem associated with the Camassa-Holm equation

CHRISTER BENNEWITZ

INTRODUCTION

Associated with the Camassa-Holm (CH) equation (see [1])

$$u_t - u_{txx} + 3uu_x + 2\varkappa u_x = 2u_x u_{xx} + uu_{xxx},$$

where $x \in \mathbb{R}$, $t \ge 0$ and $\varkappa$ is a parameter, there is the spectral problem

$$-y'' + \tfrac{1}{4}y = \lambda(\varkappa + w(\cdot, t))y, \tag{1}$$

where $w = u - u_{xx}$ and t is viewed as a parameter. There are complications in copying the scattering-inverse scattering approach for the KdV equation to this situation. In particular, an interesting feature of the CH equation is the presence of wave breaking. It is known, however, that this can only occur if $\varkappa + w(x, 0)$ is not of one sign. Standard spectral theory, however, considers (1) in an L^2-space with weight $\varkappa + w$, which is only possible if the weight is of one sign.

One may instead use $H^1(\mathbb{R})$ as the Hilbert space for (1), provided with a slightly modified scalar product

$$\langle y, z\rangle = \int_{\mathbb{R}} (y'\overline{z'} + \tfrac{1}{4}y\overline{z}), \qquad \|y\| = \sqrt{\langle y, y\rangle}.$$

A simple scaling argument shows that one need only consider the cases $\varkappa = 0$ and $\varkappa = 1$. Consider now the case $\varkappa = 1$. There is then a scattering theory for (1), with standard decay assumptions on w. It was proved in [2] [3] that all eigen-values and the transmission coefficient are conserved quantities under the CH flow, and that the reflection coefficient and the normalization constants for the eigenfunctions evolve in a simple, explicit way. Unfortunately, no inverse scattering theory is available unless $1 + w \geq 0$. In this case one may transform (1) to a standard Schrödinger equation, and use the inverse scattering theory then available. This was carried out by Constantin [2] and Lenells [3].

On the other hand, there is a complete spectral theory for (1), and some inverse spectral theory. This is of some use in the case $\varkappa = 0$, as we shall see.

SPECTRAL THEORY

We sketch a general spectral theory for equations of the form

$$-(py')' + qy = \lambda wy \quad \text{in } [0, b), \tag{2}$$

where $p \geq 0$, $q \geq 0$ and $1/p$, q and w are all in $L^1_{\text{loc}}[0, b)$. For simplicity also assume $\operatorname{supp} w = [0, b)$. We study the equation in the completion $\mathcal{H}$ of $C^1_0(0, b)$ with respect to the norm-square $\|y\|^2 = \int_0^b (p|y'|^2 + q|y|^2)$. Let $\varphi(x, \lambda)$ be the solution of (2) with initial data $\varphi(0, \lambda) = 0$, $p\varphi'(0, \lambda) = 1$. There then exists a uniquely determined positive measure $d\rho$ on $\mathbb{R}$, called the *spectral measure*, some of the properties of which are as follows. Let L^2_ρ be the Hilbert space of functions $\hat{y}$ measurable $(d\rho)$ and such that $\int |\hat{y}|^2 \, d\rho < \infty$. Given $y \in \mathcal{H}$ the integral $\hat{y}(t) = \int_0^b (py'\varphi'(\cdot, t) + qy\varphi(\cdot, t))$ converges in L^2_ρ and gives a unitary map $\mathcal{F} : \mathcal{H} \ni y \mapsto \hat{y} \in L^2_\rho$, the *generalized Fourier transform* for (2). The spectrum of the operator corresponding to (2) is $\operatorname{supp} d\rho$, eigenvalues corresponding to point-masses in the measure. We have the following inverse spectral theorem.

Theorem 0.1. Suppose the interval $[0, b)$ and the coefficients p and q are given. Then the spectral measure determines the coefficient w uniquely.

If the spectrum is discrete with eigenvalues λ_n and we define the normalization constants $c_n = \|\varphi(\cdot, \lambda_n)\|^{-2}$, then $d\rho = \sum c_n \delta_{\lambda_n}$. So, in this case knowing the spectral measure is equivalent to knowing all eigenvalues and normalization constants.

For a brief indication of the proof of the theorem, assume that two coefficients w and $\tilde{w}$ give the same spectral measure, and let $\mathcal{U} = \tilde{\mathcal{F}}^{-1}\mathcal{F}$, $\mathcal{F}$ and $\tilde{\mathcal{F}}$ being the generalized Fourier transforms associated with w and $\tilde{w}$ respectively. Then $\mathcal{U}$ is a unitary operator on $\mathcal{H}$, and we are done if we can prove that it is the identity. It is not hard to see that this follows if we can prove that $\mathcal{U}$ preserves supports. To see that it does, one may use a generalization of the classical Paley-Wiener theorem, valid for the generalized Fourier transforms used here.

APPLICATION TO THE CAMASSA-HOLM EQUATION

We consider (1) with $\varkappa = 0$ on $(-\infty, \infty)$, where w is locally integrable. We may transform this problem using a *Liouville transform*, introducing new independent and dependent variables $\xi(x) = e^{-x}$ and $\tilde{y}(\xi) = e^{-x/2}y(x)$. If $y \in \mathcal{H}$ we obtain

$$\int_{\mathbb{R}} (|y'|^2 + \tfrac{1}{4}|y|^2) = \int_0^\infty |\tilde{y}'|^2.$$

The equation (1) is transformed to $-\tilde{y}'' = \lambda\tilde{w}\tilde{y}$ where $\tilde{w}(\xi) = e^{2x}w(x)$. The spectral theory sketched above applies if $\tilde{w}$ is integrable near 0. This translates into the requirement that $e^x w(x)$ is integrable near $+\infty$.

Assuming $(1+|x|)w(x) \in L^1(\mathbb{R})$ and $\varkappa = 0$, the spectrum of (1) is discrete and the equation has a solution $f_+(x,\lambda)$ asymptotic to $e^{-x/2}$ at $+\infty$ for any λ. It is easy to see that f_+ transforms to φ, and f_+ will be in $\mathcal{H}$ precisely if $\lambda = \lambda_n$ is an eigen-value. Define the corresponding normalization constant

$$c_n = \Big(\int_{\mathbb{R}} (|f_+'(\cdot,\lambda_n)|^2 + \tfrac{1}{4}|f_+(\cdot,\lambda_n)|^2)\Big)^{-1}. \tag{3}$$

Eigen-values are still conserved under the CH flow, and the normalization constants evolve according to $c_n(t) = c_n(0)\exp(-t/4\lambda_n)$. Clearly $c_n = \|\varphi(\cdot,\lambda_n)\|^{-2}$, so Theorem 0.1 gives the following theorem, which is at least a step in the direction of a scattering-inverse scattering approach for CH.

Theorem 0.2. Assume that $\varkappa = 0$ and $(1+e^x)w(x) \in L^1(\mathbb{R})$. Then (1) has discrete spectrum and the eigenvalues and normalization constants (3) determine w uniquely.

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Algebro-Geometric Solutions of the KdV and Camassa-Holm equation

HELGE HOLDEN

(joint work with Fritz Gesztesy)

THE KDV HIERARCHY

To construct the KdV hierarchy, one assumes u to be a smooth function on $\mathbb{R}$ (or meromorphic in $\mathbb{C}$) in the stationary context or a smooth function on $\mathbb{R}^2$ in the time-dependent case, and one introduces the recursion relation for some functions f_ℓ of u by

$$f_0 = 1, \quad f_{\ell,x} = -(1/4)f_{\ell-1,xxx} + uf_{\ell-1,x} + (1/2)u_x f_{\ell-1}, \quad \ell \in \mathbb{N}. \tag{1}$$

Given the recursively defined sequence $\{f_\ell\}_{\ell\in\mathbb{N}_0}$ (whose elements turn out to be differential polynomials with respect to u, defined up to certain integration constants) one defines the Lax pair of the KdV hierarchy by

$$L = -\frac{d^2}{dx^2} + u, \tag{2}$$

$$P_{2n+1} = \sum_{\ell=0}^{n} \Big(f_{n-\ell}\frac{d}{dx} - \frac{1}{2}f_{n-\ell,x}\Big)L^{\ell}. \tag{3}$$

The commutator of P_{2n+1} and L then reads

$$[P_{2n+1}, L] = 2f_{n+1,x}, \tag{4}$$

using the recursion (1). Introducing a deformation (time) parameter $t_n \in \mathbb{R}$, $n \in \mathbb{N}_0$ into u, the *KdV hierarchy* of nonlinear evolution equations is then defined by imposing the *Lax commutator relations*

$$\frac{d}{dt_n}L - [P_{2n+1}, L] = 0, \tag{5}$$

for each $n \in \mathbb{N}_0$. By (4), the latter are equivalent to the collection of evolution equations

$$\mathrm{KdV}_n(u) = u_{t_n} - 2f_{n+1,x}(u) = 0, \quad n \in \mathbb{N}_0. \tag{6}$$

Explicitly,

$$\begin{aligned} \mathrm{KdV}_0(u) &= u_{t_0} - u_x = 0,\\ \mathrm{KdV}_1(u) &= u_{t_1} + \tfrac{1}{4}u_{xxx} - \tfrac{3}{2}uu_x - c_1u_x = 0,\\ \mathrm{KdV}_2(u) &= u_{t_2} - \tfrac{1}{16}u_{xxxxx} + \tfrac{5}{8}uu_{xxx} + \tfrac{5}{4}u_xu_{xx} - \tfrac{15}{8}u^2u_x\\ &\quad + c_1\big(\tfrac{1}{4}u_{xxx} - \tfrac{3}{2}uu_x\big) - c_2u_x = 0, \quad \text{etc.,} \end{aligned} \tag{7}$$

represent the first few equations of the time-dependent KdV hierarchy.

We construct a special class of explicitly defined solutions given by the Its–Matveev formula

$$u(x,t_n) = \Lambda_0 - 2\partial_x^2 \ln(\theta(\underline{A} + \underline{B}x + \underline{C}_r t_n)), \tag{8}$$

Here $\Lambda_0, \underline{A}, \underline{B}, \underline{C}_r$ are all constants, and θ is Riemann's theta function. Observe that the argument in the theta-function is linear both in space and time.

THE CAMASSA–HOLM HIERARCHY

The Camassa–Holm (CH) equation reads

$$4u_t - u_{xxt} - 2uu_{xxx} - 4u_xu_{xx} + 24uu_x = 0, \quad (x,t) \in \mathbb{R}^2 \tag{9}$$

(chosing a scaling of x, t that's convenient for our purpose), with u representing the fluid velocity in x-direction. Actually, (9) represents the limiting case $\kappa \to 0$ of the general Camassa–Holm equation,

$$4v_t - v_{xxt} - 2vv_{xxx} - 4v_xv_{xx} + 24vv_x + 4\kappa v_x = 0, \quad \kappa \in \mathbb{R},\ (x,t) \in \mathbb{R}^2. \tag{10}$$

However, in our formalism the general Camassa–Holm equation (10) just represents a linear combination of the first two equations in the CH hierarchy and hence we consider without loss of generality (9) as the first nontrivial element of the Camassa–Holm hierarchy. Alternatively, one can transform

$$v(x,t) \mapsto u(x,t) = v(x - (\kappa/2)t, t) + (\kappa/4) \tag{11}$$

and thereby reduce (10) to (9).

We start by formulating the basic polynomial setup. One defines $\{f_\ell\}_{\ell\in\mathbb{N}_0}$ recursively by

$$f_0 = 1,$$
$$f_{\ell,x} = -2\mathcal{G}\big(2(4u - u_{xx})f_{\ell-1,x} + (4u_x - u_{xxx})f_{\ell-1}\big), \quad \ell \in \mathbb{N}, \tag{12}$$

where $\mathcal{G}$ is given by

$$\mathcal{G}\colon L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R}), \quad (\mathcal{G}v)(x) = \frac{1}{4}\int_{\mathbb{R}} dy\, e^{-2|x-y|}v(y), \quad x \in \mathbb{R},\ v \in L^\infty(\mathbb{R}). \tag{13}$$

One observes that $\mathcal{G}$ is the resolvent of minus the one-dimensional Laplacian at energy parameter equal to -4, that is,

$$\mathcal{G} = \left(-\frac{d^2}{dx^2} + 4\right)^{-1}. \tag{14}$$

The first coefficient reads

$$f_1 = -2u + c_1, \tag{15}$$

where c_1 is an integration constant. Subsequent coefficients are nonlocal with respect to u. At each level a new integration constant, denoted by c_ℓ, is introduced. Moreover, we introduce coefficients $\{g_\ell\}_{\ell\in\mathbb{N}_0}$ and $\{h_\ell\}_{\ell\in\mathbb{N}_0}$ by

$$g_\ell = f_\ell + \frac{1}{2}f_{\ell,x}, \quad \ell \in \mathbb{N}_0, \tag{16}$$

$$h_\ell = (4u - u_{xx})f_\ell - g_{\ell+1,x}, \quad \ell \in \mathbb{N}_0. \tag{17}$$

Explicitly, one computes

$$\begin{aligned}
f_0 &= 1,\\
f_1 &= -2u + c_1,\\
f_2 &= 2u^2 + 2\mathcal{G}\big(u_x^2 + 8u^2\big) + c_1(-2u) + c_2,\\
g_0 &= 1,\\
g_1 &= -2u - u_x + c_1,\\
g_2 &= 2u^2 + 2uu_x + 2\mathcal{G}\big(u_x^2 + u_xu_{xx} + 8uu_x + 8u^2\big)\\
&\quad + c_1(-2u - u_x) + c_2,\\
h_0 &= 4u + 2u_x,\\
h_1 &= -2u_x^2 - 4uu_x - 8u^2\\
&\quad - 2\mathcal{G}\big(u_xu_{xxx} + u_{xx}^2 + 2u_xu_{xx} + 8uu_{xx} + 8u_x^2 + 16uu_x\big)\\
&\quad + c_1(4u + 2u_x), \text{ etc.}
\end{aligned} \tag{18}$$

Next one introduces the 2×2 matrix U by

$$U(z, x) = \begin{pmatrix} -1 & 1 \\ z^{-1}(4u(x) - u_{xx}(x)) & 1 \end{pmatrix}, \quad x \in \mathbb{R}, \tag{19}$$

and for each $n \in \mathbb{N}_0$ the following 2×2 matrix V_n by

$$V_n(z,x) = \begin{pmatrix} -G_n(z,x) & F_n(z,x) \\ z^{-1}H_n(z,x) & G_n(z,x) \end{pmatrix}, \quad n \in \mathbb{N}_0,\ z \in \mathbb{C} \setminus \{0\},\ x \in \mathbb{R}, \tag{20}$$

assuming F_n, G_n, and H_n to be polynomials of degree n with respect to z and C^∞ in x. Postulating the zero-curvature condition

$$-V_{n,x}(z,x) + [U(z,x), V_n(z,x)] = 0, \tag{21}$$

one finds

$$F_{n,x}(z,x) = 2G_n(z,x) - 2F_n(z,x), \tag{22}$$

$$zG_{n,x}(z,x) = (4u(x) - u_{xx}(x))F_n(z,x) - H_n(z,x), \tag{23}$$

$$H_{n,x}(z,x) = 2H_n(z,x) - 2(4u(x) - u_{xx}(x))G_n(z,x). \tag{24}$$

From (22)–(24) one infers that

$$\frac{d}{dx}\det(V_n(z,x)) = -\frac{1}{z}\frac{d}{dx}\Big(zG_n(z,x)^2 + F_n(z,x)H_n(z,x)\Big) = 0, \tag{25}$$

and hence

$$zG_n(z,x)^2 + F_n(z,x)H_n(z,x) = Q_{2n+1}(z), \tag{26}$$

where the polynomial Q_{2n+1} of degree $2n+1$ is x-independent. Actually it turns out that it is more convenient to define

$$R_{2n+2}(z) = zQ_{2n+1}(z) = \prod_{m=0}^{2n+1}(z - E_m), \quad E_0 = 0,\ E_1, \ldots, E_{2n+1} \in \mathbb{C} \tag{27}$$

so that (26) becomes

$$z^2G_n(z,x)^2 + zF_n(z,x)H_n(z,x) = R_{2n+2}(z). \tag{28}$$

Next one makes the ansatz that F_n, H_n, and G_n are polynomials of degree n, related to the coefficients f_ℓ, h_ℓ, and g_ℓ by

$$F_n(z,x) = \sum_{\ell=0}^{n} f_{n-\ell}(x)z^\ell, \tag{29}$$

$$G_n(z,x) = \sum_{\ell=0}^{n} g_{n-\ell}(x)z^\ell, \tag{30}$$

$$H_n(z,x) = \sum_{\ell=0}^{n} h_{n-\ell}(x)z^\ell. \tag{31}$$

Insertion of (29)–(31) into (22)–(24) then yields the recursion relations (12)–(13) and (16) for f_ℓ and g_ℓ for $\ell = 0, \ldots, n$. For fixed $n \in \mathbb{N}$ we obtain the recursion (17) for h_ℓ for $\ell = 0, \ldots, n-1$ and

$$h_n = (4u - u_{xx})f_n. \tag{32}$$

(When $n=0$ one directly gets $h_0=(4u-u_{xx})$.) Moreover, taking $z=0$ in (28) yields

$$f_n(x)h_n(x) = -\prod_{m=1}^{2n+1} E_m. \tag{33}$$

In addition, one finds

$$h_{n,x}(x) - 2h_n(x) + 2(4u(x)-u_{xx}(x))g_n(x) = 0, \quad n\in\mathbb{N}_0. \tag{34}$$

Using the relations (16) and (32) permits one to write (34) as

$$\text{s-CH}_n(u) = (u_{xxx}-4u_x)f_n - 2(4u-u_{xx})f_{n,x} = 0, \quad n\in\mathbb{N}_0. \tag{35}$$

Varying $n\in\mathbb{N}_0$ in (35) then defines the stationary CH hierarchy. We record the first few equations explicitly,

$$\begin{aligned}
&\text{s-CH}_0(u) = u_{xxx}-4u_x = 0,\\
&\text{s-CH}_1(u) = -2uu_{xxx}-4u_xu_{xx}+24uu_x + c_1(u_{xxx}-4u_x) = 0,\\
&\text{s-CH}_2(u) = 2u^2u_{xxx}-8uu_xu_{xx}-40u^2u_x + 2(u_{xxx}-4u_x)\mathcal{G}\big(u_x^2+8u^2\big)\\
&\qquad - 8(4u-u_{xx})\mathcal{G}\big(u_xu_{xx}+8uu_x\big)\\
&\qquad + c_1(-2uu_{xxx}-4u_xu_{xx}+24uu_x) + c_2(u_{xxx}-4u_x) = 0, \text{ etc.}
\end{aligned} \tag{36}$$

Next, we turn to the time-dependent CH hierarchy. Introducing a deformation parameter $t_n\in\mathbb{R}$ into u (replacing $u(x)$ by $u(x,t_n)$), the definitions (19), (20), and (29)–(31) of U, V_n, and F_n, G_n, and H_n, respectively, still apply. The corresponding zero-curvature relation reads

$$U_{t_n}(z,x,t_n) - V_{n,x}(z,x,t_n) + [U(z,x,t_n),V_n(z,x,t_n)] = 0, \quad n\in\mathbb{N}_0, \tag{37}$$

which results in the following set of equations

$$\begin{aligned}
&4u_{t_n}(x,t_n) - u_{xxt_n}(x,t_n) - H_{n,x}(z,x,t_n) + 2H_n(z,x,t_n)\\
&-2(4u(x,t_n)-u_{xx}(x,t_n))G_n(z,x,t_n) = 0,
\end{aligned} \tag{38}$$

$$F_{n,x}(z,x,t_n) = 2G_n(z,x,t_n) - 2F_n(z,x,t_n), \tag{39}$$

$$zG_{n,x}(z,x,t_n) = (4u(x,t_n)-u_{xx}(x,t_n))F_n(z,x,t_n) - H_n(z,x,t_n). \tag{40}$$

Inserting the polynomial expressions for F_n, H_n, and G_n into (39) and (40), respectively, first yields recursion relations (12) and (16) for f_ℓ and g_ℓ for $\ell=0,\dots,n$. For fixed $n\in\mathbb{N}$ we obtain from (38) the recursion (17) for h_ℓ for $\ell=0,\dots,n-1$ and

$$h_n = (4u-u_{xx})f_n. \tag{41}$$

(When $n=0$ one directly gets $h_0=(4u-u_{xx})$.) In addition, one finds

$$\begin{aligned}
&4u_{t_n}(x,t_n) - u_{xxt_n}(x,t_n) - h_{n,x}(x,t_n) + 2h_n(x,t_n)\\
&- 2(4u(x,t_n)-u_{xx}(x,t_n))g_n(x,t_n) = 0, \quad n\in\mathbb{N}_0.
\end{aligned} \tag{42}$$

Using relations (16) and (41) permits one to write (42) as

$$\text{CH}_n(u) = 4u_{t_n} - u_{xxt_n} + (u_{xxx}-4u_x)f_n - 2(4u-u_{xx})f_{n,x} = 0, \quad n\in\mathbb{N}_0. \tag{43}$$

Varying $n \in \mathbb{N}_0$ in (43) then defines the time-dependent CH hierarchy. We record the first few equations explicitly,

$$\begin{aligned}
&\mathrm{CH}_0(u) = 4u_{t_0} - u_{xxt_0} + u_{xxx} - 4u_x = 0,\\
&\mathrm{CH}_1(u) = 4u_{t_1} - u_{xxt_1} - 2uu_{xxx} - 4u_x u_{xx} + 24uu_x + c_1(u_{xxx} - 4u_x) = 0,\\
&\mathrm{CH}_2(u) = 4u_{t_2} - u_{xxt_2} + 2u^2u_{xxx} - 8uu_xu_{xx} - 40u^2u_x\\
&\qquad + 2(u_{xxx} - 4u_x)\mathcal{G}\big(u_x^2 + 8u^2\big) - 8(4u - u_{xx})\mathcal{G}\big(u_xu_{xx} + 8uu_x\big)\\
&\qquad + c_1(-2uu_{xxx} - 4u_xu_{xx} + 24uu_x) + c_2(u_{xxx} - 4u_x) = 0, \text{ etc.}
\end{aligned} \tag{44}$$

We show the analogue of the Its–Matveev formula for the CH hierarchy. Here we find

$$u(x, t_n) = A + \sum_{j=1}^{n} U_j \frac{\partial}{\partial w_j} \ln\left(\frac{\theta\big(\underline{z}(P_{\infty_+}, \underline{\hat{\mu}}(x, t_r)) + \underline{w}\big)}{\theta\big(\underline{z}(P_{\infty_-}, \underline{\hat{\mu}}(x, t_r)) + \underline{w}\big)}\right)\Bigg|_{\underline{w}=0}. \tag{45}$$

Here $(U_1, \dots, U_n)$ is a constant, and

$$\underline{\hat{z}}(P, \underline{Q}) = B(P) + \underline{\hat{\alpha}}(\mathcal{D}_{\underline{Q}}), \tag{46}$$

where $B(P)$ is a constant, and $\underline{\hat{\alpha}}$ is the Abel map, and $\mathcal{D}_{\underline{Q}}$ is a divisor at $\underline{Q}$. Finally, $\underline{\hat{\mu}}$ is the set of solutions of the Dubrovin equations. All constants can be explicitly computed in terms of quantities of a hyperelliptic curve. Unfortunately, the argument inside the theta function is not linear in the space and time variable.

Extensive background information and complete details as well as references to the early literature can be found in [1].

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Lie Groups and Mechanics: an introduction

Boris Kolev

Euler equation of a rigid body

In classical mechanics, a *material system* (Σ) in the ambient space $\mathbb{R}^3$ is described by a *positive measure* μ on $\mathbb{R}^3$ with compact support. This measure is called the *mass distribution* of (Σ).

In the *Lagrangian formalism of Mechanics*, a *motion* of a material system is described by a smooth path φ^t of *embeddings* of the *reference state* $\Sigma = Supp(\mu)$ in the ambient space. A material system (Σ) is *rigid* if each map φ is the restriction to Σ of an isometry g of the Euclidean space $\mathbb{R}^3$.

In what follows, we are going to study the motions of a rigid body (Σ) such that $\Sigma = Supp(\mu)$ spans the 3 space. In that case, the manifold of all possible configurations of (Σ) is completely described by the group $\mathcal{D}_3$ of orientation-preserving isometries of $\mathbb{R}^3$.

Although the physically meaningful rigid body mechanics is in dimension 3, we will not use this peculiarity in order to distinguish easier the main underlying concepts. Hence, in what follows, we will study the motion of an n-dimensional rigid body. To avoid heavy computations, we will restrain our study to motions of a rigid body having a fixed point. In these circumstances, the configuration space reduces to the group $SO(n)$ of isometries which fix a point.

The Lie algebra $\mathfrak{so}(n)$ of $SO(n)$ is the space of all skew-symmetric $n \times n$ matrices[1]. There is a canonical inner product, the so-called *Killing form*

$$\langle \Omega_1, \Omega_2 \rangle = -\frac{1}{2} \operatorname{tr}(\Omega_1 \Omega_2)$$

which permit us to identify $\mathfrak{so}(n)$ with its dual space $\mathfrak{so}(n)^*$. For x and y in $\mathbb{R}^n$, we define

$$L^*(x,y)(\Omega) = (\Omega\, x) \cdot y, \quad \Omega \in \mathfrak{so}(n)$$

which is skew-symmetric in x, y and thus defines a linear map

$$L^* : \bigwedge^2 \mathbb{R}^n \to \mathfrak{so}(n)^* \ .$$

This map is injective and defines therefore an isomorphism between $\mathfrak{so}(n)^*$ and $\bigwedge^2 \mathbb{R}^n$, which have the same dimension. We let $L(x, y)$ be the corresponding element of $\mathfrak{so}(n)$ (using the Killing form).

The location of a point a of the body Σ is described by the column vector r of its coordinates in the frame $\Re_0$. At time t, this point occupies a new position $r(t) = g(t)r$, where $g(t)$ is an element of the group $SO(3)$ and its velocity is given by $\mathbf{v}(a, t) = \dot{g}(t)\, r$. The kinetic energy K of the body Σ at time t is defined by

$$K(t) = \frac{1}{2} \int_\Sigma \|\mathbf{v}(a,t)\|^2 \ d\mu = \frac{1}{2} \int_\Sigma \|\dot{g}\, r\|^2 \ d\mu = \frac{1}{2} \int_\Sigma \|\Omega r\|^2 \ d\mu \tag{1}$$

where $\Omega = g^{-1}\, \dot{g}$ lies in the Lie algebra $\mathfrak{so}(n)$.

Lemma 1. *We have $K = -\frac{1}{2} \operatorname{tr}(\Omega J \Omega)$, where J is the symmetric matrix with entries*

$$J_{ij} = \int_\Sigma x_i x_j \ d\mu \ .$$

The kinetic energy K is therefore a positive quadratic form on the Lie algebra $\mathfrak{so}(n)$. To K, a linear operator $A : \mathfrak{so}(n) \to \mathfrak{so}(n)$, called the *inertia tensor* or the *inertia operator*, is associated by means of the relation

$$K = \frac{1}{2} \langle A(\Omega), \Omega \rangle \,, \qquad \Omega \in \mathfrak{so}(n).$$

More precisely, this operator is given by

$$A(\Omega) = J\Omega + \Omega J = \int_\Sigma \left(\Omega\, r r^t + r r^t \Omega \right) \, d\mu \ . \tag{2}$$

[1] In dimension 3, we generally identify the Lie algebra $\mathfrak{so}(3)$ with $\mathbb{R}^3$ endowed with the Lie bracket given by the cross product $\omega_1 \times \omega_2$.

The *angular momentum* of the rigid body is defined by the following 2-vector

$$\mathcal{M}(t) = \int_\Sigma (gr) \wedge (\dot{g}r)\; d\mu\ .$$

Lemma 2. *We have* $L(\mathcal{M}) = gA(\Omega)g^{-1}$.

If there are no external actions on the body, the spatial angular momentum is a constant of the motion,

$$\frac{d\mathcal{M}}{dt} = 0\ . \tag{3}$$

Coupled with the relation $L(\mathcal{M}) = gA(\Omega)g^{-1}$, we deduce that

$$A(\dot\Omega) = A(\Omega)\Omega - \Omega A(\Omega) \tag{4}$$

which is the generalization in n dimensions of the traditional *Euler equation*. Notice that if we let $M = A(\Omega)$, this equation can be rewritten as

$$\dot M = [M, \Omega\,]\,. \tag{5}$$

GENERAL ARNOLD-EULER EQUATION

A *Riemannian* or *pseudo-Riemannian* metric on a Lie group G is left invariant if it is preserved under every left shift L_g, that is,

$$\langle X_g, Y_g\,\rangle_g = \langle L_h\, X_g, L_h\, Y_g\,\rangle_{hg}\,, \qquad g, h \in G.$$

A left-invariant metric is uniquely defined by its restriction to the tangent space to the group at the unity and hence by a quadratic form on the Lie algebra of the group, $\mathfrak{g}$. To such a quadratic form on $\mathfrak{g}$, correspond a symmetric operator $A : \mathfrak{g} \to \mathfrak{g}^{\,*}$ defined by

$$\langle \xi, \omega\,\rangle = (A\xi, \omega\,) = (A\omega, \xi\,)\,, \qquad \xi, \omega \in \mathfrak{g}\,.$$

The operator A is called the *inertia operator*. It can be extended to a left-invariant tensor $A_g : T_gG \to T_gG^*$ defined by $A_g = L^*_{g^{-1}} A L_{g^{-1}}$.

The geodesics of the metric are defined as extremals of the *Lagrangian*

$$\mathcal{L}(g) = \int K\left(g(t), \dot g(t)\right)\, dt \tag{6}$$

where

$$K(X) = \frac{1}{2}\,\langle X_g, X_g\,\rangle_g = \frac{1}{2}\,(A_g\, X_g, X_g\,)_g$$

is called the *kinetic energy* or *energy functional*.

If $g(t)$ is a geodesic, the velocity $\dot g(t)$ can be translated to the identity via left or right shifts and we obtain two elements of the Lie algebra $\mathfrak{g}$,

$$\omega_L = L_{g^{-1}}\dot g, \qquad \omega_R = R_{g^{-1}}\dot g,$$

called the *left angular velocity*, respectively the *right angular velocity*. Letting $m = A_g\,\dot g \in T_gG^*$, we define the *left angular momentum* m_L and the *right angular momentum* m_R by

$$m_L = L^*_g m \in \mathfrak{g}^*, \qquad m_R = R^*_g m \in \mathfrak{g}^*.$$

Between these four elements, we have the relations

$$\omega_R = Ad_g\, \omega_L, \quad m_R = Ad_g^* m_L, \quad m_L = A\, \omega_L.$$

The invariance of the energy with respect to left translations leads to the existence of a *momentum map* $\mu : TG \to \mathfrak{g}^*$ defined by

$$\mu((g,\dot g))(\xi) = \frac{\partial K}{\partial \dot g}\, Z_\xi = \langle \dot g, R_g\, \xi\, \rangle_g = (m, R_g\, \xi\,) = \big(R_g^*\, m, \xi\,\big) = m_R(\xi),$$

where Z_ξ is the right-invariant vector field generated by $\xi \in \mathfrak{g}$. According to Noether's theorem, this map is constant along a geodesic, that is

$$\frac{dm_R}{dt} = 0.$$

As we did in the special case of the group $SO(n)$, using the relation $m_R = Ad_g^*\, m_L$ and computing the time derivative, we obtain the *Arnold-Euler equation*

$$\frac{dm_L}{dt} = ad^*_{\omega_L}\, m_L. \tag{7}$$

Using $\omega_L = A^{-1}\, m_L$ and the bilinear operator B defined by

$$\langle [a,b\,]\,, c\,\rangle = \langle B(c,a), b\,\rangle\,, \quad a,b,c \in \mathfrak{g},$$

equation (7) can be rewritten as an evolution equation on the Lie algebra

$$\frac{d\omega_L}{dt} = B(\omega_L, \omega_L)\,.$$

Well-posedness results for the generalized Benjamin-Ono equation with arbitrary large initial data

LUC MOLINET

(joint work with Francis Ribaud)

Abstract. We prove new local well-posedness results for the generalized Benjamin-Ono equation (GBO) $\partial_t u + \mathcal{H}\partial_x^2 u + u^k \partial_x u = 0$, $k \geq 2$. By combining a gauge transformation with dispersive estimates we establish the local well-posedness of (GBO) in $H^s(I\!R)$ for $s \geq 1/2$ if $k \geq 5$, $s > 1/2$ if $k = 2, 4$ and $s \geq 3/4$ if $k = 3$. Moreover we prove that in all these cases the flow map is locally Lipschitz on $H^s(I\!R)$.

PRESENTION OF THE PROBLEM

This work is devoted to the study of the local well-posedness problem for the generalized Benjamin-Ono equation

$$\text{(GBO)} \qquad \begin{cases} \partial_t u + \mathcal{H}\partial_x^2 u \pm u^k \partial_x u = 0\ , \ (t,x) \in I\!R \times I\!R\ , \\ u(0,x) = u_0(x)\ , \end{cases}$$

where $\mathcal{H}$ is the Hilbert transform defined by

$$\mathcal{H}(f)(x) = -i \int_{-\infty}^{+\infty} e^{ix\xi} \operatorname{sgn}(\xi) \hat{f}(\xi)\, d\xi \ ,$$

and $k \geq 2$ is an integer.

The Benjamin-Ono equation (k=1) arises as a model for long internal gravity waves in deep stratified fluids, see [2], and have been studied in a large amount of works. When $k \geq 2$, (GBO) is an infinite dimensional Hamiltonian system (for $k = 1$ it is even formally completely integrable) and possesses the following invariant quantities :

$$I(u) = \int_{-\infty}^{+\infty} u(t,x)\, dx\,, \quad M(u) = \int_{-\infty}^{+\infty} u^2(t,x)\, dx\,,$$

and

$$E(u) = \int_{-\infty}^{+\infty} \Big(\frac{1}{2} |D_x^{1/2} u(t,x)|^2 \mp \frac{1}{(k+1)(k+2)} u(t,x)^{k+2} \Big)\, dx \ \ (energy) \ .$$

One of the challenging problem about this family of equations is probably to establish a well-posedness result in the energy space $H^{1/2}(I\!\!R)$.

Recall that the Cauchy problem for the Benjamin-Ono equation (k=1) has been shown to be locally well-posed in $H^s(I\!\!R)$ for $s \geq 3$ in [19], $s > 3/2$ in [8], [1] and later on for $s \geq 3/2$ in [18]. These results have been extended to global ones by using conservation laws. Recently, by establishing dispersive estimates for the non homogeneous linearized equation, H. Koch and N. Tzvetkov [14] and then C. Kenig and K. Koenig [9] have improved these local well-posedness results in $H^s(I\!\!R)$ to respectively $s > 5/4$ and $s > 9/8$. More recently, using a gauge transformation and standard dispersive estimates, T. Tao [20] has gone down to $H^1(I\!\!R)$. It is worth noticing that all these results have been obtained by compactness methods. Moreover, it has been proved in [17] that, for all $s \in I\!\!R$, the flow-map $u_0 \mapsto u(t)$ is not of class C^2 at the origin in $H^s(I\!\!R)$ which implies that it is not possible to obtain well-posedness results in $H^s(I\!\!R)$ for the Benjamin-Ono equation by contraction methods. In this direction, H. Koch and N. Tzvetkov [15] have recently proved that this flow-map is even not locally uniformly continuous in $H^s(I\!\!R)$.

Now, concerning the case $k \geq 2$, the local well-posedness of (GBO) is also known in $H^s(I\!\!R)$ for $s > 3/2$, see [8], [1], [13]. Recently, using the approach developped in [14], C. Kenig and K. Koenig [9] have shown the local well-posedness of (GBO) in $H^1(I\!\!R)$ for $k = 2$ (note that only the continuity of the flow-map is established). Unfortunately, this approach does not seem to permit to go below $H^1(I\!\!R)$ due to the weakness of the smoothing effect of the associated free evolution. On the other hand, in the context of small initial data, C. Kenig, G. Ponce and L. Vega [13] have proved local well-posedness results for (GBO) in $H^s(I\!\!R)$ by a Picard iterative sheme on the integral equation. This denotes of course a strong difference with the case $k = 1$ and implies the real analiticity of the flow-map in a neighborhood of the origin. Very recently, these results have been improved by the authors in [16] where it is proven that, for small initial data, (GBO) is locally well-posed in $H^s(\mathbb{R})$ as soon as

$$\begin{cases} s > 1/2 \text{ if } k = 2\,, \\ s > 1/3 \text{ if } k = 3\,, \\ s > s_k \text{ if } k \geq 4\,, \end{cases}$$

and globally well-posed as soon as

$$\begin{cases} s \geq 1/2 \text{ if } k = 3\,, \\ s > s_k \text{ if } k \geq 4\,, \end{cases}$$

where $s_k = 1/2 - 1/k$ is the critical scaling Sobolev index. Moreover these results are almost sharp for $k \neq 3$: in [16] we prove that for $k = 2$ and $k \geq 4$, the flow map is not respectively of class C^3 below $H^{1/2}(\mathbb{R})$ and of class C^{k+1} below $H^{s_k}(\mathbb{R})$ at the origin. Note that the above results imply the global well-posedness in the energy space $H^{1/2}(\mathbb{R})$ for small initial data when $k \geq 3$.

It is worth recalling that the dispersion of the free evolution $V(t)$ of (GBO) is just sufficient to recover the lost derivative in the nonlinear term but does not seem to permit to get a contraction factor for T small when estimating the operator

$$\mathcal{G} \;:\; u \mapsto V(t)u_0 - \int_0^t V(t-t')\partial_x(u^{k+1}(t'))\,dt'$$

in the appropriate resolution space. This explains the smallness assumption on the initial data in [13] and [16]. In this sense (GBO) seems to be a limit case for the balance between dispersion and derivative nonlinearity of order one.

In this work we improve the existing local well-posedness results in the case of arbitrary large initial data. As mentioned above, the aim is to reach the energy space $H^{1/2}(\mathbb{R})$. This will be achieved for $k \geq 5$. More precisely, we prove that (GBO) is locally well-posed in $H^s(\mathbb{R})$ as soon as

$$\begin{cases} s > 1/2 \text{ if } k = 2, 4\,, \\ s \geq 3/4 \text{ if } k = 3\,, \\ s \geq 1/2 \text{ if } k \geq 5\,. \end{cases}$$

Moreover we show that in all these cases, in a sharp contrast with the case $k = 1$, the flow-map is locally Lipschitz. This has to be anderstood as a stability result for (GBO) when $k \geq 2$.

To establish our results, inspired by the recent work [20], we introduce a gauge transform w of u a smooth solution of (GBO) and derive a dispersive equation satisfied by w. Using dispersive estimates we will be able to get a positive power of T in front of the Duhamel part when estimating w in our resolution space. Next, rewriting (GBO) with the help of w, we obtain the desired estimate on the solution u. Our results follow then by regularizing the initial data and passing to the limit on smooth solutions to (GBO).

1. Main results

Let us state our main result.

Theorem 1. *For any $u_0 \in H^s(I\!R)$ with*

$$\begin{cases} s > 1/2 \ \textit{if} \ k = 2,4\,, \\ s \geq 3/4 \ \textit{if} \ k = 3\,, \\ s \geq 1/2 \ \textit{if} \ k \geq 5\,, \end{cases}$$

there exists $T = T_k^s(\|u_0\|_{H^s}) > 0$ with $T_k^s(\alpha) \nearrow \infty$ as $\alpha \searrow 0$, and a unique solution u to (GBO) satisfying

$$u \in C([0,T]; H^s(I\!R)) \cap X_T^s\ .$$

Moreover, for the class of s defined above, the flow-map is Lipschitz on every bounded set of $H^s(I\!R)$.

Remark 1.1. Actually we prove that $T = T_k(\|u_0\|_{H^{s(k)}})$ where

$$s(k) = \begin{cases} 1/2+ \ \text{ if } \ k = 2,4\,, \\ 3/4 \ \text{ if } \ k = 3\,, \\ 1/2 \ \text{ if } \ k \geq 5\,. \end{cases}$$

Remark 1.2. Theorem 1 yields a global existence result for the solutions to the following (GBO) equation

$$\partial_t u + \mathcal{H}\partial_x^2 u - u^k \partial_x u = 0\ , \tag{1}$$

where k is an odd integer greater or equal to 5. Indeed, the energy is then given by

$$E(u) = \frac{1}{2}\int_{-\infty}^{+\infty} |D_x^{1/2} u(t,x)|^2\,dx + \frac{1}{(k+1)(k+2)}\int_{-\infty}^{+\infty} u(t,x)^{k+2}\,dx$$

and thus, for $k \geq 5$ odd, Theorem 1.1 combining with the conservation of $E(u)$ leads to the global well-posedness of (1) in $H^s(I\!R)$, $s \geq 1/2$. Note that for $k \geq 2$ with the reverse sign in front of the nonlinear term, numerical simulations suggest that blow-up in finite time can occur for large initial data [4].

Remark 1.3. Following the approach developped in this work with some additional technical points, one can certainly improve the results of Theorem 1 to $s > s_k = 1/2 - 1/k$ at least for $k \geq 5$ large enough. This would be in some sense optimal since it is shown in [3] that the flow-map cannot be uniformly continuous in $H^s(I\!R)$ for $s = s_k$.

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About the "loss of regularity" for hyperbolic problems

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Mathematics Subject Classification 2000: 35L15, 35L80, 35S30, 35B45.

In this lecture we will study hyperbolic problems with quite different goals from the first point of view. It turns out that these problems have common features which are described in the following table:

strictly hyperbolic theory with non-Lipschitz continuous coefficients	$\longleftrightarrow$	$L_p - L_q$ decay estimates for wave equations with bounded time-dependent coefficients
$\updownarrow$		$\updownarrow$
weakly hyperbolic theory	$\longleftrightarrow$	$L_p - L_q$ decay estimates for wave equations with increasing time-dependent coefficients

The question for $L_p - L_q$ decay estimates is related with the question for global in time small data solutions for the Cauchy problem for nonlinear wave equations like

$$u_{tt} - \triangle u = f(u_t, \nabla u, \nabla u_t, \nabla^2 u), \;\; u(0,x) = \varepsilon\phi(x), \; u_t(0,x) = \varepsilon\psi(x).$$

The goal is to prove under suitable assumptions that for all $\varepsilon \in (0, \varepsilon_0(\phi,\psi)]$ there exists a global (in time) small data solution. One of the key tools is the so-called Strichartz' decay estimate

$$E(u)(t)|_{L_q} \leq C(1+t)^{-\frac{n-1}{2}(\frac{1}{p}-\frac{1}{q})} E(u)(0)|_{W_p^{N_p}}$$

on the conjugate line $2 \leq q \leq \infty, \;\; 1/p + 1/q = 1$ for solutions of the Cauchy problem for classical wave equations, where $N_p > n\Big(\frac{1}{p} - \frac{1}{q}\Big)$. Generalizing such type of estimates (with $-\frac{n}{2}$ instead of $-\frac{n-1}{2}$ in the decay rate) to Klein-Gordon equations or damped wave equations (with an additional term $-\frac{1}{2}$ in the decay rate of the latter case coming from the dissipation itself) one can show the global existence of small data solutions for

$$u_{tt} - \triangle u + m^2 u = f(u_t, \nabla u, \nabla u_t, \nabla^2 u), \;\; u(0,x) = \varepsilon\phi(x), \; u_t(0,x) = \varepsilon\psi(x), \; m > 0;$$
$$u_{tt} - \triangle u + u_t = f(u_t, \nabla u, \nabla u_t, \nabla^2 u), \;\; u(0,x) = \varepsilon\phi(x), \; u_t(0,x) = \varepsilon\psi(x).$$

In general one can find such $L_p - L_q$ decay estimates for solutions of partial differential equations (or systems) with constant coefficients.

For this reason the author asked if one can generalize such estimates to solutions for wave equations with time dependent coefficients like

$$u_{tt} - a(t) \triangle u + m(t)u + b(t)u_t = 0.$$

Here one can use the WKB-method and construct explicit representations of solutions. The dependence of coefficients on spatial variables brings essential difficulties, e.g. the global existence (in time) of phase functions in the FIO-representations.
The above model is to general, one should assume some more structure of the coefficients.

1.case: wave equations with weak dissipation
The model under consideration is

$$u_{tt} - \triangle u + b(t)u_t = 0.$$

Under the main assumptions $b' < 0, \lim_{t\to\infty} b(t) = 0$ we have a complete picture from wave to damped wave equations which reads in the following form (we only describe the decay rates):

- $b \in L_1(\mathbb{R}^+)$: scattering results with the free wave equation,
- $b(t) \sim (t \log t)^{-1}$ for large t: hyperbolic decay rate $-\frac{n-1}{2}(\frac{1}{p} - \frac{1}{q})$ and a term coming from the dissipation itself, such dissipations are *not effective*,
- $b(t) = \mu(1+t)^{-1}$, $\mu > 0$: critical case $\mu = 2$ gives the best $L_2 - L_2$ decay rate $(1+t)^{-1}$, here the decay rate changes from the hyperbolic one (small μ) to the parabolic one $-\frac{n}{2}(\frac{1}{p} - \frac{1}{q})$ (large μ),
- $b(t) \sim t^{-1}\log t$ for large t: parabolic decay rate $-\frac{n}{2}(\frac{1}{p} - \frac{1}{q})$ and a term coming from the dissipation itself, such dissipations are *effective*.

Question: What are the relations in the 3-d case between the influence of dissipation (effective or non effective) on $L_p - L_q$ decay estimates and assumptions to the asymptotic behavior of the nonlinearity $f = f(u_t, \nabla u, \nabla u_t, \nabla^2 u)$ in 0?

2.case: general model There exist several difficulties:
- There exists an interplay between oscillating behavior and increasing behavior of coefficients.
- An interplay between $a = a(t)$ and $m(t)$ decides if the *mass term is effective*. In such a case it should be included into the phase function. This gives difficulties to develop a stationary phase method.
- An interplay between $a = a(t)$ and $b(t)$ decides if the *dissipation term is effective*.

One can prove the following results:
1. *Let us consider the model problem*

$$u_{tt} - \exp(2t^\alpha)(2 + \sin t)^2 \triangle u = 0, \;\; u(0,x) = \phi(x), \;\; u_t(0,x) = \psi(x).$$

Then the following Strichartz' type estimate holds with some regularity $W_p^{N_p}$*:*

$$E(u)(t)|_{L_q} \leq C\Big(1 + \int_0^t \exp \tau^\alpha d\tau\Big)^{s_0 - \frac{n-1}{2}(\frac{1}{p} - \frac{1}{q})} E(u)(0)|_{W_p^{N_p}},$$

where s_0=ε *sufficiently small,* s_0 *is a positive constant,* $s_0 = \infty$ *(no* $L_p - L_q$ *decay estimate) if* $\alpha > \frac{1}{2}$*,* $\alpha = \frac{1}{2}$*,* $\alpha < \frac{1}{2}$ *respectively.*
2. A mass term can have an improving influence (less increasing behavior is necessary) as the next result shows.

Let us consider the model problem

$$u_{tt} - (1 + t)^2(2 + \sin t)^2(\triangle u - u) = 0, \;\; u(0,x) = \phi(x), \;\; u_t(0,x) = \psi(x).$$

Then the following Strichartz' type estimate holds with some regularity $W_p^{N_p}$*:*

$$E(u)(t)|_{L_q} \leq C(1 + t^2)^{s_0 - \frac{n}{2}(\frac{1}{p} - \frac{1}{q})} E(u)(0)|_{W_p^{N_p}},$$

where s_0 *is a positive constant.*

Are there some relations to other hyperbolic problems? **Yes!** There exist relations to *weakly hyperbolic problems* or to *strictly hyperbolic problems with non-Lipschitz coefficients*. Let us demonstrate this connection by the following result:

Let us consider the strictly hyperbolic Cauchy problem

$$u_{tt} - \sum_{k,l=1}^{n} a_{kl}(t,x) u_{x_k x_l} = f(t,x), \;\; u(0,x) = \phi(x), u_t(0,x) = \psi(x),$$

in the strip $\mathbb{R}^n \times [0,T]$. *The non-Lipschitz behavior of coefficients is described by the following conditions for all multi-indices* β *and all* $p \in \mathbb{N}$*:*

$$|D_t^p D_x^\beta a_{kl}(t,x)| \le C_{p\beta}\Big(\frac{1}{t}\Big(\log\frac{1}{t}\Big)\Big)^\gamma.$$

Then for large s *the energy inequality*

$$E(u)(t)|_{H^{s-s_0}} \le C_s E(u)(0)|_{H^s}$$

holds with $s_0 = 0$, $s_0{=}\varepsilon$ *sufficiently small,* s_0 *is a positive constant,* $s_0 = \infty$ *(no* C^∞ *well-posedness) if* $\gamma = 0$, $\gamma \in (0,1)$, $\gamma = 1$, $\gamma > 1$ *respectively. Moreover, there exists a parametrix in the cases* $\gamma \in [0,1]$.

Remarks:
• There exists in all those hyperbolic problems a connection between the oscillating behavior of coefficients and the "loss of regularity" (for $L_p - L_q$ decay estimates this means how the decay rate differs from the classical decay rates for the wave, Klein-Gordon, or damped wave operator). An optimal classification of oscillations can be given for all problems.
• The construction of parametrix in form of Fourierintegral operators is closely related to the construction of representation of solutions by Fourier multipliers to derive $L_p - L_q$ decay estimates.
• A careful division of the phase space into zones, a symbolic calculus for non-standard symbol classes, hierarchies of symbols, ellipticity, the construction of phase functions and amplitudes in FIO-representations, and a suitable perfect diagonalization procedure, form the main tools for the construction of parametrix.
• Counterexamples ($s_0 = \infty$ in the above results) are proved by the application of Floquet's theory.

Genesis of Solitons Arising from Individual Flows of the Camassa-Holm Hierarchy

Enrique Loubet

The present work offers a detailed account of the large time development of the velocity profile v run by a single "individual" Hamiltonian flow of the Camassa-Holm (CH) hierarchy, the Hamiltonian employed being the invariant $\mathrm{H} = 1/\lambda$, where λ is *any* of the bound states of the associated spectral problem: $(\frac{1}{4} - D^2)(f) = \lambda m f$, with "mass" potential $m \equiv v - v''$. The flow may be expressed as in $\partial m/\partial t = [mD + Dm](f^2) = 1/(2\lambda)D(1 - D^2)(f^2)$, or more simply, as $\partial v/\partial t = 1/(2\lambda)D(f^2)$. Unlike the formation of the soliton train that is produced by Korteweg-de Vries (KdV) $\partial V/\partial t = 3V\partial V/\partial X - \frac{1}{2}\partial^3 V/\partial X^3$, which accounts, except for the reflectionless potential V, only for the part of the total energy ascribed to the bound states of the associated spectral problem $(-D^2 + V)(F) = \lambda F$, the deficiency being carried by the evanescent radiation corresponding to the continuous spectrum; for *summable* m, CH has *only* bound states λ_n , $n \in \mathbb{Z} - \{0\}$, each of which characterizes the speed=amplitude of the associated individual soliton $\mathcal{S}_n(t,x) \equiv 1/(2\lambda_n)e^{-|x-t/(2\lambda_n)|}$. They embody respectively an energy $\frac{1}{2}\int[(\mathcal{S}_n')^2 + \mathcal{S}_n^2] = 1/(4\lambda_n^2)$, and all these individual pieces add up to the whole: $\mathrm{H}_{\mathrm{CH}} \equiv \frac{1}{2}\int m\mathrm{G}[m] = \sum 1/(4\lambda_n^2)$ where $\mathrm{G} \equiv (1-D^2)^{-1}$, so here nothing is lost. And indeed, the present investigation confirms this:

Let m be summable and odd, having the signature of x, and consider the individual flow based upon $\mathrm{H} = 1/\lambda$ with $\lambda > 0$. With the help of a private "Lagrangian" scale determined by $\overline{x}^\bullet = -f^2(t,\overline{x})$ and $\overline{x}(0,x) = x$; the updated velocity profile $(e^{t\mathbb{X}_\mathrm{H}}v(0,\cdot))(\overline{x}) \equiv v(t,\overline{x}(t,x))$ is found to shape itself like the soliton

$$\mathcal{S}_\lambda(t,x) = 1/(2\lambda)e^{-|x-t/(2\lambda)|}$$

escaping to $-\infty$ as $t \uparrow +\infty$, leaving behind a "residual" $v(+\infty,\overline{x}(+\infty,x))$ having the same spectrum as the one attatched to the initial $v(0,x)$ except that λ is excised. Doubtless, the map induced by the large time asymptotics $v(0,x) \mapsto v(+\infty,\overline{x}(+\infty,x))$, is some counterpart of the standard Darboux transformation for removing/adding the bottom bound state for KdV, with the difference now that you need not proceed in such orderly fashion. I did not succeed in casting such correspondence in the form of an "addition" as in [1], but it should be closely connected to that circle of ideas.

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